THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271
FINAL EXAMINATION, WINTER 2007
TIME: 3 HOURS

NAME________________________________ID____________Section________

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Total (max. 80)	

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 9 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(78, 41)$. Also use the algorithm to find integers x and y such that $\gcd(78, 41) = 78x + 41y$.

(b) Use part (a) to find an inverse for 41 modulo 78; that is, find an integer $a \in \{0, 1, \ldots, 77\}$ so that $41a \equiv 1 \bmod 78$.

[11] 2. Let G be the graph with vertices $\{0, 1, 2, 3, 4, 5\}$ and edges defined by: for all $a, b \in \{0, 1, 2, 3, 4, 5\}$, a and b are adjacent if and only if $a + b$ is odd.

(a) Draw the graph G.

(b) Is G a complete graph? Explain.

(c) Is G a complete bipartite graph? Explain.

(d) Does G have a Hamiltonian circuit? Explain.

[9] 3. Let $\mathbb{Z}^+$ be the set of all positive integers, and define the relation $\mathcal{R}$ on $\mathcal{P}(\mathbb{Z}^+)$ (the power set of $\mathbb{Z}^+$) by:

for any subsets A, B of $\mathbb{Z}^+$, $A\mathcal{R}B$ if and only if $B \subseteq A$.

(a) Is $\mathcal{R}$ reflexive? Symmetric? Transitive? Give reasons.

(b) Suppose that $A = \{1, 2, 3, \ldots, 271\}$. Find the number of sets $B \in \mathcal{P}(\mathbb{Z}^+)$ such that $A\mathcal{R}B$. Explain.

[12] 4. Let $\mathcal{F}_3$ denote the set of all functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$.
(a) Find the number of functions $f \in \mathcal{F}_3$ such that $f(1) = 3$. How many of these are one-to-one? Explain.

(b) Find the number of functions $f \in \mathcal{F}_3$ such that $(f \circ f)(1) = 3$.

(c) Find two functions f and g in $\mathcal{F}_3$ so that f is one-to-one, g is onto, and $(f \circ g)(1) = 3$.

(d) Find the number of ordered pairs (f, g) of functions in $\mathcal{F}_3$ so that $(f \circ g)(1) = 3$.

[12] 5. (a) Prove or disprove:

(i) For all sets A and B, if $A \subseteq B \cup \{1, 2, 3\}$ then $A - \{1, 2, 3\} \subseteq B$.

(ii) For all sets A and B, if $A \cup \{1, 2, 3\} \subseteq B$ then $A \subseteq B - \{1, 2, 3\}$.

(iii) For all sets A and B, if $A \subseteq B - \{1, 2, 3\}$ then $A \cup \{1, 2, 3\} \subseteq B$.

(b) Write out (as simply as possible) the *negation* of the statement in (a) part (iii) above.

[6] 6. Prove or disprove:

(a) $\forall a \in \mathbb{Z}\ \exists b \in \mathbb{Z}$ such that $a+b$ is prime.

(b) $\exists b \in \mathbb{Z}$ such that $\forall a \in \mathbb{Z}$, $a+b$ is prime.

[5] 7. (a) Let n be a positive integer. For integers a and b, define what $a \equiv b \bmod n$ means.

(b) For each positive integer n, we know that the relation $\equiv \bmod n$ defined in part (a) is an equivalence relation on $\mathbb{Z}$. Suppose that n is a positive integer such that 9 and 19 are in the same equivalence class of $\equiv \bmod n$. Find all possible values of n. Explain.

[9] 8. Suppose a graph G has exactly five vertices a, b, c, d, e and has adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

(a) Draw the graph G.

(b) How many subgraphs does G have with exactly two vertices? Explain. Simplify your answer.

(c) Draw a graph H with six vertices a, b, c, d, e, f so that H has an Euler circuit and so that G is a subgraph of H. Explain.

(d) Draw a **tree** with five vertices a, b, c, d, e which is a subgraph of G.

[8] 9. Let $\mathcal{S}$ be the statement

$$\frac{1!}{2^1} + \frac{2!}{2^2} + \cdots + \frac{n!}{2^n} \leq \frac{(n+1)!}{2^{n+1}} \, .$$

(a) Show that $\mathcal{S}$ is true when $n = 1$ but is **false** when $n = 2$.

(b) Prove by mathematical induction that $\mathcal{S}$ is true for all integers $n \geq 4$.

$\overline{[80]}$