THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271 (L01, L02)
FINAL EXAMINATION, WINTER 2008
TIME: 3 HOURS

NAME____________________________________ID_____________Section______

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8	
Total (max. 80)	

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 9 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(102, 43)$. Also use the algorithm to find integers x and y such that $\gcd(102, 43) = 102x + 43y$.

(b) Use part (a) to find an inverse a for 43 modulo 102 so that $0 \leq a \leq 101$; that is, find an integer $a \in \{0, 1, \ldots, 101\}$ so that $43a \equiv 1 \pmod{102}$.

[11] 2. $\mathbb{Z}$ is the set of all integers. Let S be the statement:

$$\forall n \in \mathbb{Z}, \text{ if } 2|n \text{ and } 6|n, \text{ then } 12|n.$$

(a) Is S true? Give a proof or disproof.

(b) Write out the *converse* of statement S. Is it true or false? Explain.

(c) Write out the *contrapositive* of statement S. Is it true or false? Explain.

[12] 3. Let $\mathbb{R}$ be the set of all real numbers, and define the relation R on $\mathbb{R}$ by:

for all $x, y \in \mathbb{R}$, xRy if and only if xy is an integer.

(a) Is R reflexive? Symmetric? Transitive? Give reasons.

(b) Prove or disprove: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}$ so that xRy.

(c) Prove or disprove: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}$ so that xRy but $x \not R\ (y+1)$ (that is, x is not related to $y+1$).

[9] 4. Let $\mathcal{F}$ denote the set of all functions from $\{1,2,3\}$ to $\{1,2,3\}$.
(a) Prove or disprove the following statement: $\forall f \in \mathcal{F}$, if $(f \circ f)(1) = 1$ then $f(1) = 1$.

(b) Write out the *negation* of the statement in part (a).

(c) Find the *number* of functions $f \in \mathcal{F}$ such that $(f \circ f)(1) = 1$. Explain.

(d) Find the number of one-to-one onto functions $f \in \mathcal{F}$ so that $f(1) = 2$ and $f^{-1}(2) = 3$. Explain.

[15] 5. Let $S = \{1, 2, \ldots, 10\}$. Define a relation $\mathcal{R}$ on the power set $\mathcal{P}(S)$ of all subsets of S by: for all $A, B \in \mathcal{P}(S)$, $A\mathcal{R}B$ if and only if the number of odd integers in A is equal to the number of odd integers in B.

(a) Prove that $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(S)$.

(b) Find the number of equivalence classes of $\mathcal{R}$. Explain.

(c) Find three elements in the equivalence class $[\{1, 2, 3\}]$.

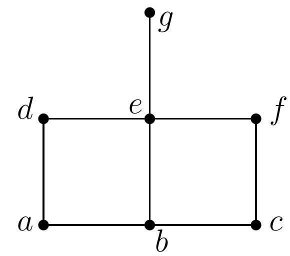
(d) Find the number of elements in the equivalence class $[\{1, 2, 3\}]$. Simplify your answer.

[7] 6. One of the following statements is true and one is false. Prove the true statement and disprove the false statement.

(a) for all sets A, B, C, D, $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

(d) for all sets A, B, C, D, $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$.

[12] 7. Let G be the graph shown at the right.



(a) Find a spanning tree of G.

(b) Does G have an Euler circuit? Explain. If no, add one new edge to G so that the new graph does have an Euler circuit, and draw the new graph here.

(c) Does G have a Hamiltonian circuit? Explain. If no, add one new edge to G so that the new graph does have a Hamiltonian circuit, and draw the new graph here.

(d) Find a subgraph of G which is isomorphic to the complete bipartite graph $K_{2,2}$.

[6] 8. Define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_0 = 1,\ a_1 = 6, \text{ and } a_n = 3a_{n-1} - 2a_{n-2} - 5 \text{ for all integers } n \geq 2.$$

Use strong mathematical induction (or well ordering) to prove that $a_n = 5n + 1$ for all integers $n \geq 0$.

[80]