THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271 (L01)
FINAL EXAMINATION, FALL 2009
TIME: 3 HOURS

NAME________________________________ ID____________ Section______

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| Total (max. 80) | |

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 10 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(98, 31)$. Also use the algorithm to find integers $x$ and $y$ such that $\gcd(98, 31) = 98x + 31y$.

(b) Use part (a) to find an inverse $a$ for 31 modulo 98 so that $0 \leq a \leq 97$; that is, find an integer $a \in \{0, 1, \ldots, 97\}$ so that $31a \equiv 1 \pmod{98}$.

[11] 2. $\mathbb{R}$ is the set of all real numbers. Let $S$ be the statement:

$$\forall a \in \mathbb{R}, \text{ if } a \text{ is rational then } a + \sqrt{2} \text{ is irrational.}$$

(a) Prove that $S$ is true. You may use that $\sqrt{2}$ is irrational and that the sum and difference of rational numbers is rational.

(b) Write out the *converse* of statement $S$. Is it true or false? Explain.

(c) Write out the *contrapositive* of statement $S$. Is it true or false? Explain.

[12] 3. Let $S = \{1,2,3,4,5,6\}$, and define the relation $\mathcal{R}$ on the power set $\mathcal{P}(S)$ of all subsets of $S$ by:

$$\text{for all } A, B \in \mathcal{P}(S),\ A\mathcal{R}B \text{ if and only if } 1 \in A \cap B.$$

(a) Is $\mathcal{R}$ reflexive? Symmetric? Transitive? Give reasons.

(b) Find the number of sets $B \in \mathcal{P}(S)$ so that $\{1\}\mathcal{R}B$.

(c) Find the number of sets $B \in \mathcal{P}(S)$ which have exactly three elements and so that $\{1\}\mathcal{R}B$. Simplify your answer.

[9] 4. Let $\mathcal{F}$ denote the set of all functions from $\{1,2,3,4,5\}$ to $\{1,2,3,4\}$.
(a) Find the number of functions $f \in \mathcal{F}$ so that $f(1) = f(2)$.

(b) Find the number of functions $f \in \mathcal{F}$ satisfying: there exists some $x \in \{1,2,3,4,5\}$ so that $f(x) = 1$.

(c) Find the number of functions $f \in \mathcal{F}$ satisfying: for all $x \in \{1,2,3,4,5\}$, $f(x)$ is odd if and only if $x$ is even.

[4] 5. Prove the following statement: for all sets $A$, $B$ and $C$, if $B \subseteq C$ then $A \times B \subseteq A \times C$.

[6] 6. (a) *Disprove* the following statement: for all sets $A$ and $B$, $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.
(Here $\mathcal{P}(X)$ denotes the power set of the set $X$.)

(b) Write out the *negation* of the statement in part (a).

[7] 7. $\mathbb{Z}$ is the set of all integers. One of the following two statements is true and one is false. Prove the true statement and disprove the false statement.

(a) $\forall m \in \mathbb{Z}\ \exists n \in \mathbb{Z}$ so that $m - n = 271$.

(b) $\forall m \in \mathbb{Z}\ \exists n \in \mathbb{Z}$ so that $m/n = 271$.

[8] 8. Define a relation $R$ on the set $\mathbb{Z}$ of all integers by: for all $a, b \in \mathbb{Z}$, $aRb$ if and only if $a^2 = b^2$.

(a) Prove that $R$ is an equivalence relation on $\mathbb{Z}$.

(b) Find all the elements in the equivalence class $[271]$.

[9] 9. (a) Draw a connected graph with 6 vertices and 8 edges which is not Eulerian. (Explain why it is not Eulerian.)

(b) Draw a tree with 6 vertices so that the degree of every vertex is odd.

(c) Draw a graph whose adjacency matrix is $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. (Label the vertices of your graph and explain their connection with the matrix.)

[6] 10. Prove using mathematical induction (or well ordering) that $3 \mid (7^n + 2)$ for all integers $n \geq 0$.

$\overline{[80]}$