MATH 271 ASSIGNMENT 1 SOLUTIONS

1. For each true statement below, give a proof. For each false statement below, write out its negation and prove that. [*Hint*: you may use Exercise 16, page 147.]

   (a) $\forall q \in \mathbb{Q}\ \exists r \in \mathbb{Q}$ so that $q + r \in \mathbb{Z}$.

   (b) $\forall q \in \mathbb{Q}\ \exists r \in \mathbb{Q}$ so that $q + r \notin \mathbb{Z}$.

   (c) $\forall q \in \mathbb{Q}\ \exists r \in \mathbb{Q}$ so that $qr \in \mathbb{Z}$.

   (d) $\forall q \in \mathbb{Q}\ \exists r \in \mathbb{Q}$ so that $qr \notin \mathbb{Z}$.

   (e) $\forall q \in \mathbb{Q}\ \exists r \in \mathbb{Q}$ so that $q + r \notin \mathbb{Z}$ and $qr \in \mathbb{Z}$.

   (a) This statement is **true**. Here is a proof. Let q be an arbitrary rational number. We want to find some rational number r so that $q+r$ is an integer. Let $r = -q$. By Exercise 16, page 147, r is rational. Also, $q + r = q + (-q) = 0$ which is an integer. Done. □

   (b) This statement is **true**. Here is a proof. Let q be an arbitrary rational number. We want to find some rational number r so that $q + r$ is *not* an integer. We do this in two cases.

   Case 1: Suppose that q is not an integer. Then we let $r = 0$ (which is rational). Then $q + r = q + 0 = q$ is not an integer, so we are done with this case.

   Case 2: Suppose that q is an integer. Then we let $r = 1/2$ (which is rational). Then $q + r = q + \frac{1}{2}$ is an integer plus $1/2$, which is never an integer, so we are done with this case as well. This finishes the proof of the problem. □

   (c) This statement is **true**. Here is a proof. Let q be an arbitrary rational number. We want to find some rational number r so that qr is an integer. Let $r = 0$ (which is rational). Then $qr = q \cdot 0 = 0$ which is an integer. Done. □

   (d) This statement is **false**. The negation of this statement is

   $$\exists q \in \mathbb{Q} \text{ so that } \forall r \in \mathbb{Q}, qr \in \mathbb{Z}.$$

   We want to prove the negation, which means that we want to find some rational number q so that, for all rational numbers r, qr is an integer. An example (in fact the only example) is $q = 0$ (which is rational). Then for every rational r (even every real r), $qr = 0 \cdot r = 0$ which is an integer. □

   Note. You could prove that this statement is true if q is restricted to *nonzero* rationals.

   (e) This statement is **false**. The negation of this statement is

   $$\exists q \in \mathbb{Q} \text{ so that } \forall r \in \mathbb{Q}, q + r \in \mathbb{Z} \ \textbf{ or } \ qr \notin \mathbb{Z}.$$

   We want to prove the negation, which means that we want to find some rational number q so that, for all rational numbers r, either $q + r$ is an integer or qr is not an integer. An example is $q = 1$ (which is rational). Then for every rational r, $q + r = 1 + r$ while $qr = 1 \cdot r = r$. So if r is not an integer, then $qr = r$ is not an integer, which is what we want. On the other hand, if r is an integer, then $q + r = 1 + r$ is also an integer, which again is what we want. $\square$

Note. You could also use $q = -1$. But these are the only values of q for which the original statement is false. Try to prove that

$$\forall q \in \mathbb{Q} - \{1, -1\}\ \exists r \in \mathbb{Q} \text{ so that } q + r \notin \mathbb{Z} \text{ and } qr \in \mathbb{Z}.$$

If you get a proof, show it to your professor or TA.

2. In this question you may assume without proof that every integer is either even or odd (but not both) and also that consecutive integers have opposite parity, but otherwise use only the definitions of even and odd integers.

   (a) Prove using contradiction or the contrapositive: $\forall a \in \mathbb{Z}$, if a is odd then $a/2 \notin \mathbb{Z}$.

   (b) Write out the *converse* of the statement in (a). Is it true? Explain.

   (c) Let $\mathcal{S}$ be the statement: $\forall a \in \mathbb{Z}$, if a is odd then $\lfloor a/2 \rfloor$ is odd. If $\mathcal{S}$ is true, prove it. If $\mathcal{S}$ is false, write out its negation and prove that.

   (d) Prove or disprove: $\forall a \in \mathbb{Z}$, if a is odd then $\lfloor a/2 \rfloor$ is odd or $\lceil a/2 \rceil$ is odd.

   (e) Write out the *contrapositive* of the statement in (d). Is it true? Explain.

   (a) *Proof using contradiction.* Let a be an arbitrary odd integer. We want to prove that $a/2$ is not an integer. So suppose that $a/2$ is an integer. Then we can write $a/2 = k$ for some integer k. This means that $a = 2k$, which means by definition that a is even. But this contradicts the assumption that a is odd (since no integer can be both odd and even). Therefore $a/2$ cannot be an integer. Done. $\square$

   Proof using the contrapositive. The contrapositive is: $\forall a \in \mathbb{Z}$, if $a/2 \in \mathbb{Z}$ then a is even. Let a be an arbitrary integer so that $a/2$ is an integer. Then $a/2 = k$ for some integer k. Thus $a = 2k$, which means by definition that a is even. This proves the contrapositive, therefore the original statement must be true too. $\square$

   (b) The converse is: $\forall a \in \mathbb{Z}$, if $a/2 \notin \mathbb{Z}$ then a is odd.

   The converse is **true**. Here is a proof. Let a be an arbitrary integer so that $a/2$ is not an integer. We want to prove that a is odd. We again use contradiction. Suppose that a is not odd, which means (since a is an integer) that a must be even. This means that $a = 2k$ for some integer k, and so $a/2 = k$, an integer. But this contradicts the assumption that $a/2$ is not an integer. Thus a must be odd. $\square$

   Note: we could also write out the contrapositive of the converse and prove it instead.

   (c) The statement $\mathcal{S}$ is **false**. The negation of $\mathcal{S}$ is $\mathcal{S}$: $\exists a \in \mathbb{Z}$ so that a is odd and $\lfloor a/2 \rfloor$ is even. Here is a proof of the negation. An example is $a = 1$, which is an odd integer, and $\lfloor a/2 \rfloor = \lfloor 1/2 \rfloor = 0$ which is even. $\square$

   (d) This statement is **true**. Here is a proof. Let a be an arbitrary odd integer. Then $a = 2k + 1$ for some $k \in \mathbb{Z}$. So $a/2 = k + 1/2$, and since k is an integer this must mean that $\lfloor a/2 \rfloor = k$ and $\lceil a/2 \rceil = k+1$. Since k and $k+1$ are consecutive integers, one of them must be odd. So one of $\lfloor a/2 \rfloor$ and $\lceil a/2 \rceil$ must be odd. $\square$

Here is an alternate proof. Let a be an arbitrary odd integer. Then by part (a) of this question, $a/2$ is not an integer. Thus $a/2$ cannot be equal to either $\lfloor a/2 \rfloor$ or $\lceil a/2 \rceil$, both of which must be integers. In fact we get that $\lfloor a/2 \rfloor < a/2 < \lceil a/2 \rceil$, where $\lfloor a/2 \rfloor$ and $\lceil a/2 \rceil$ must be consecutive integers. Therefore, since consecutive integers have opposite parity, one of $\lfloor a/2 \rfloor$ and $\lceil a/2 \rceil$ must be odd (and the other even). $\square$

(e) The contrapositive is: $\forall a \in \mathbb{Z}$, if $\lfloor a/2 \rfloor$ is even **and** $\lceil a/2 \rceil$ is even then a is even. It is true because it is equivalent to the original statement, which is true.

3. (a) Let N be your U of C ID number. Use the Euclidean algorithm to calculate $d = \gcd(N, 271)$. Then use your calculations to find integers x and y so that $Nx + 271y = d$.

(b) Suppose a certain student's ID number M satisfies

$$\gcd(M, 2010) > \gcd(M, 271) > 1.$$

Find all possible values for $\gcd(M, 2010)$. Be sure to explain your reasoning. [*Note*: both 271 and 67 are prime.]

(c) Suppose that the ID number M from part (b) lies between 10020000 and 10030000. Find M. Be sure to explain your reasoning.

(a) Let's do it for the hypothetical student number $N = 12341234$. The Euclidean algorithm gives:

$$\begin{aligned}
12341234 &= 45539 \cdot 271 + 165 && (\text{so } 165 = 12341234 - 45539 \cdot 271)\\
271 &= 1 \cdot 165 + 106 && (\text{so } 106 = 271 - 165)\\
165 &= 1 \cdot 106 + 59 && (\text{so } 59 = 165 - 106)\\
106 &= 1 \cdot 59 + 47 && (\text{so } 47 = 106 - 59)\\
59 &= 1 \cdot 47 + 12 && (\text{so } 12 = 59 - 47)\\
47 &= 3 \cdot 12 + 11 && (\text{so } 11 = 47 - 3 \cdot 12)\\
12 &= 1 \cdot 11 + 1 && (\text{so } 1 = 12 - 11)\\
11 &= 11 \cdot 1,
\end{aligned}$$

so $\gcd(12341234, 271) = \mathbf{1}$, the last nonzero remainder.

Now, starting with the second-last equation above, solving it for the gcd 1, and plugging in the remainders one by one from the earlier equations, we get:

$$\begin{aligned}
1 &= 12 - 11\\
&= 12 - (47 - 3 \cdot 12) = 4 \cdot 12 - 47\\
&= 4 \cdot (59 - 47) - 47 = 4 \cdot 59 - 5 \cdot 47\\
&= 4 \cdot 59 - 5 \cdot (106 - 59) = 9 \cdot 59 - 5 \cdot 106\\
&= 9 \cdot (165 - 106) - 5 \cdot 106 = 9 \cdot 165 - 14 \cdot 106\\
&= 9 \cdot 165 - 14 \cdot (271 - 165) = 23 \cdot 165 - 14 \cdot 271\\
&= 23 \cdot (12341234 - 45539 \cdot 271) - 14 \cdot 271 = 23 \cdot 12341234 - 1047411 \cdot 271.
\end{aligned}$$

So $x = 23$ and $y = -1047411$ in this case.

We could also use the table method taught in class. Then we would get

row operation		12341234	271	what this row says
	12341234	1	0	$12341234 = 1 \cdot 12341234 + 0 \cdot 271$
	271	0	1	$271 = 0 \cdot 12341234 + 1 \cdot 271$
$R1 - 45539R2$	165	1	-45539	$165 = 1 \cdot 12341234 + (-45539) \cdot 271$
$R2 - R3$	106	-1	45540	$106 = (-1) \cdot 12341234 + 45540 \cdot 271$
$R3 - R4$	59	2	-91079	$59 = 2 \cdot 12341234 + (-91079) \cdot 271$
$R4 - R5$	47	-3	136619	$47 = (-3) \cdot 12341234 + 136619 \cdot 271$
$R5 - R6$	12	5	-227698	$12 = 5 \cdot 12341234 + (-227698) \cdot 271$
$R6 - 3R7$	11	-18	819713	$11 = (-18) \cdot 12341234 + 819713 \cdot 271$
$R7 - R8$	**1**	23	-1047411	$1 = 23 \cdot 12341234 + (-1047411) \cdot 271$

Thus from the last row we see that

$$\gcd(12341234, 271) = \mathbf{1} = 23 \cdot 12341234 + (-1047411) \cdot 271.$$

(b) Let $\gcd(M, 271) = d$ and $\gcd(M, 2010) = e$. Since $d > 1$, and d must divide into 271, and 271 is prime, d must equal 271. Thus M must be a multiple of 271, and also $e > 271$. e must also divide into 2010, whose prime factorization is $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Divisors of 2010 are any products of these factors. We need a product for e which is bigger than 271, so we need to include 67 and at least one other factor among the factors $2, 3, 5$ of 2010. There are five possibilities for e:

$$67 \times 5 = \mathbf{335}, \qquad 67 \times 3 \times 2 = \mathbf{402}, \qquad 67 \times 5 \times 2 = \mathbf{670},$$

$$67 \times 5 \times 3 = \mathbf{1005}, \qquad 67 \times 2 \times 3 \times 5 = \mathbf{2010}.$$

(c) M must be a multiple of e. M is also a multiple of 271 (which is relatively prime to any of the choices for e), so M is a multiple of $271e$. Trying the five possibilities for e from part (b), we get:

- if $e = 335$, then $271e = 90785$.
- if $e = 402$, then $271e = 108942$.
- if $e = 670$, then $271e = 181570$.
- if $e = 1005$, then $271e = 272355$.
- if $e = 2010$, then $271e = 544710$.

We also need M to be between 10020000 and 10030000, and the only multiple of any of the above five numbers that lies in this range is $108942 \cdot 92 = 10022664$. (We need only check that no multiple of 90785 lies in this range, because the other three numbers are all multiples of 90785.) Therefore the answer is $M = \mathbf{10022664}$.