MATH 271 ASSIGNMENT 3 SOLUTIONS

1. Prove or disprove the following statements. Use the element method (pages 269 and 279).

   (a) For all sets A, B and C, $A \cap (B - C) = \emptyset$ if and only if $A \cap B \subseteq C$.

   (b) For all sets A, B and C, if $A \cup (B - C) = \emptyset$, then $A \cup B \subseteq C$.

   (c) For all sets A, B and C, if $A \cup B \subseteq C$, then $A \cup (B - C) = \emptyset$.

   (d) For all sets A, B and C, $A \cup (B - C) = \emptyset$ if and only if $A \cup B \subseteq C$.

   (a) This statement is **true**. Here is a proof.

   First we prove that

   $$\text{if } A \cap (B - C) = \emptyset, \text{ then } A \cap B \subseteq C.$$

   Assume that $A \cap (B - C) = \emptyset$. We want to prove that $A \cap B \subseteq C$. Let $x \in A \cap B$ be arbitrary. This means that $x \in A$ and $x \in B$. We want to prove that $x \in C$. We do this by contradiction. Suppose that $x \notin C$. Then, since $x \in B$, we get that $x \in B - C$. But also $x \in A$, so $x \in A \cap (B - C)$. But this is a contradiction since $A \cap (B - C) = \emptyset$. Therefore x must be in C, so $A \cap B \subseteq C$.

   Second, we prove that

   $$\text{if } A \cap B \subseteq C, \text{ then } A \cap (B - C) = \emptyset.$$

   Assume that $A \cap B \subseteq C$. We want to prove that $A \cap (B - C) = \emptyset$, and we also do this by contradiction (as on page 279). Assume that $A \cap (B - C) \neq \emptyset$, so there is some element $x \in A \cap (B - C)$. This means $x \in A$ and $x \in B - C$, which says $x \in B$ and $x \notin C$. Since $x \in A$ and $x \in B$, we get $x \in A \cap B$. Since $A \cap B \subseteq C$, this means $x \in C$. But this contradicts $x \notin C$. Therefore $A \cap (B - C)$ must be empty.

   (b) This statement is **true**. Here is a proof.

   Assume that $A \cup (B - C) = \emptyset$. We want to prove that $A \cup B \subseteq C$. Choose an arbitrary element $x \in A \cup B$. We want to prove that $x \in C$. $x \in A \cup B$ says that either $x \in A$ or $x \in B$. But if $x \in A$, then $x \in A \cup (B - C)$, which is impossible since $A \cup (B - C) = \emptyset$. So it must be true that $x \in B$. Now if $x \notin C$, we would get $x \in B - C$, which is also impossible since $A \cup (B - C) = \emptyset$. Thus we know that $x \in C$. Therefore $A \cup B \subseteq C$.

   (c) This statement is **false**. A counterexample is $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$. Then $A \cup B = \{1, 2\} \subseteq C$, but $A \cup (B - C) = \{1\} \cup \emptyset = \{1\} \neq \emptyset$.

   (d) Since (c) is false, this equivalence is **false** too.

2. In this question, let $S = \{1, 2, 3, 4\}$. Explain all answers completely, but you need not simplify your answers.

   (a) How many permutations of the power set $\mathscr{P}(S)$ are there?

   (b) How many permutations of $\mathscr{P}(S)$ are there, so that all subsets which contain the number 2 come before all subsets which don't contain 2?

(c) How many permutations of $\mathscr{P}(S)$ are there, so that no subset is ever followed by a subset of **smaller** size? For instance, you would not be allowed to put the subset $\{3\}$ after the subset $\{1, 2\}$ in your list because $\{3\}$ has smaller size than $\{1, 2\}$.

(d) Find the number of ordered pairs (B, C) where B and C are **disjoint** subsets of S. For instance, $(\{2\}, \{1\})$ is such an ordered pair, and so is $(\{1\}, \{2\})$, but $(\{2\}, \{1, 2\})$ is not since $\{2\}$ and $\{1, 2\}$ are not disjoint. [*Hint*: build such an ordered pair one element at a time.]

(a) There are $2^4 = 16$ subsets of $\{1, 2, 3, 4\}$ (that is, elements of $\mathscr{P}(S)$), so there are $\mathbf{16!}$ permutations of $\mathscr{P}(S)$.

(b) The subsets of S that don't contain 2 are just the subsets of $\{1, 3, 4\}$, so there are $2^3 = 8$ of these. So there must be $16 - 8 = 8$ subsets which do contain 2. Each permissible permutation of $\mathscr{P}(S)$ must consist of a list of the 8 subsets containing 2, followed by a list of the 8 subsets not containing 2. There are 8! ways to form each of these separate lists, so there are $8! \cdot 8! = \mathbf{(8!)^2}$ ways to list all 16 subsets.

(c) This time a permissible permutation has to start with the smallest subset (which is the empty set), followed by the subsets of size 1 in some order, then by the subsets of size 2 in some order, then by the subsets of size 3 in some order, and ending with the largest subset, namely the entire set $\{1, 2, 3, 4\}$. There are four subsets of size 1 (namely $\{1\}, \{2\}, \{3\}, \{4\}$), so there are 4! ways to list these four subsets. Similarly there are $\binom{4}{2} = 6$ subsets of size 2, and thus 6! ways to list them, and 4 subsets of size 3 and thus 4! ways to list them. Thus the total number of permutations we can form is $1 \cdot 4! \cdot 6! \cdot 4! \cdot 1 = \mathbf{(4!)^2 6!} = 414720$.

(d) For B and C to be disjoint, each element of $\{1, 2, 3, 4\}$ can belong to either B or C or neither. So we go through the elements $1, 2, 3, 4$ in this order, for each element saying whether it is in B, in C, or in neither. For instance, the choices "in B, in neither, in C, in C" would correspond to the disjoint subsets $B = \{1\}$ and $C = \{3, 4\}$. We get 3 choices for each of the four elements of $\{1, 2, 3, 4\}$, so there are $3^4 = 81$ such lists of choices, so $\mathbf{81}$ such ordered pairs (B, C).

3. (a) Find $\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2n-1}{2} + \binom{2n}{2}$ for $n = 1, 2$ and 3. *Note*: there are $2n - 1$ terms in this sum, with alternating signs, beginning and ending with $+$.

(b) Using your answers to part (a) (and more calculations if you need them), guess a simple formula for $\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2n-1}{2} + \binom{2n}{2}$ in terms of n.

(c) Use *induction* (or well ordering) to prove that your guess in part (b) is correct for all positive integers n.

(d) Give a *combinatorial proof* (for example, see pages 357 and 360) that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} = \binom{n}{3}$$

for all integers $n \geq 3$. [*Hint*: addition rule. How many 3-element subsets of $\{1, 2, \ldots, n\}$ have largest element n? How many have largest element $n - 1$? And so on.]

(a) When $n = 1$, the expression is just $\binom{2}{2} = \mathbf{1}$. When $n = 2$, the expression is

$$\binom{2}{2} - \binom{3}{2} + \binom{4}{2} = 1 - 3 + 6 = \mathbf{4}.$$

When $n = 3$, the expression is

$$\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \binom{5}{2} + \binom{6}{2} = 1 - 3 + 6 - 10 + 15 = \mathbf{9}.$$

(b) From part (a) we might guess that

$$\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2n-1}{2} + \binom{2n}{2} = n^2$$

for all integers $n \geq 1$.

(c) Here is a proof by induction of the formula in part (b).

Basis Step. When $n = 1$, we already checked the equation in part (a).

Inductive Step. Assume that the equation is true for some integer $n = k \geq 1$. We want to prove that the equation is true for the next integer $n = k + 1$. So we are assuming that

$$\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2k-1}{2} + \binom{2k}{2} = k^2, \tag{1}$$

and we want to prove that

$$\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2(k+1)-1}{2} + \binom{2(k+1)}{2} = (k+1)^2. \tag{2}$$

Add

$$-\binom{2(k+1)-1}{2} + \binom{2(k+1)}{2}$$

to both sides of equation (1). This gives

$$\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2k-1}{2} + \binom{2k}{2} - \binom{2(k+1)-1}{2} + \binom{2(k+1)}{2}$$
$$= k^2 - \binom{2(k+1)-1}{2} + \binom{2(k+1)}{2}$$

The left side is the same as the left side of (2) which we want to prove, so it means we need to prove that the right sides are equal. That is, we need to prove that

$$k^2 - \binom{2(k+1)-1}{2} + \binom{2(k+1)}{2} = (k+1)^2.$$

Well,

$$\begin{aligned}
k^2 - \binom{2(k+1)-1}{2} + \binom{2(k+1)}{2} &= k^2 - \binom{2k+1}{2} + \binom{2k+2}{2} \\
&= k^2 - \frac{(2k+1)(2k)}{2} + \frac{(2k+2)(2k+1)}{2} \\
&= k^2 + \frac{2k+1}{2}[-2k + (2k+2)] \\
&= k^2 + (2k+1) = (k+1)^2,
\end{aligned}$$

which is what we needed to prove. So by induction, the equation must be true for all integers $n \geq 1$.

(d) The number of 3-element subsets of $\{1, 2, \ldots, n\}$ is just $\binom{n}{3}$. We count these subsets again by dividing them into groups according to the largest element in each subset, and counting how many subsets there are in each group. The largest element in any 3-element subset of $\{1, 2, \ldots, n\}$ must obviously be at least 3. Thus we can partition the 3-element subsets of $\{1, 2, \ldots, n\}$ into $n-2$ groups $S_3, S_4, \ldots, S_n$, where for each integer $k \in \{3, 4, \ldots, n\}$, S_k is the set of all 3-element subsets of $\{1, 2, \ldots, n\}$ with largest element k. Now for each $k \in \{3, 4, \ldots, n\}$, if a 3-element subset of $\{1, 2, \ldots, n\}$ has largest element k, then the other two elements in the subset must belong to the $(k-1)$-element set $\{1, 2, \ldots, k-1\}$. Thus there are exactly $\binom{k-1}{2}$ such subsets; that is, S_k has exactly $\binom{k-1}{2}$ elements for each k. By the addition rule, the total number of 3-element subsets of $\{1, 2, \ldots, n\}$ must be

$$\sum_{k=3}^{n} \binom{k-1}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2}.$$

Since this must also equal $\binom{n}{3}$, we get the formula in (d).

Note: The equation in (d) is Exercise 14, page 362, where they ask for a proof by induction.