MATH 271 ASSIGNMENT 3

Due 4:00 PM Friday, March 19, 2010. Put your assignments in the appropriate wooden box (corresponding to your lecture section and last name) outside MS569. Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. Prove or disprove the following statements. Use the element method (pages 269 and 279).
   (a) For all sets A, B and C, $A \cap (B - C) = \emptyset$ if and only if $A \cap B \subseteq C$.
   (b) For all sets A, B and C, if $A \cup (B - C) = \emptyset$, then $A \cup B \subseteq C$.
   (c) For all sets A, B and C, if $A \cup B \subseteq C$, then $A \cup (B - C) = \emptyset$.
   (d) For all sets A, B and C, $A \cup (B - C) = \emptyset$ if and only if $A \cup B \subseteq C$.

2. In this question, let $S = \{1, 2, 3, 4\}$. Explain all answers completely, but you need not simplify your answers.
   (a) How many permutations of the power set $\mathscr{P}(S)$ are there?
   (b) How many permutations of $\mathscr{P}(S)$ are there, so that all subsets which contain the number 2 come before all subsets which don't contain 2?
   (c) How many permutations of $\mathscr{P}(S)$ are there, so that no subset is ever followed by a subset of **smaller** size? For instance, you would not be allowed to put the subset $\{3\}$ after the subset $\{1, 2\}$ in your list because $\{3\}$ has smaller size than $\{1, 2\}$.
   (d) Find the number of ordered pairs (B, C) where B and C are **disjoint** subsets of S. For instance, $(\{2\}, \{1\})$ is such an ordered pair, and so is $(\{1\}, \{2\})$, but $(\{2\}, \{1, 2\})$ is not since $\{2\}$ and $\{1, 2\}$ are not disjoint. [*Hint*: build such an ordered pair one element at a time.]

3. (a) Find $\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2n-1}{2} + \binom{2n}{2}$ for $n = 1, 2$ and 3. *Note*: there are $2n - 1$ terms in this sum, with alternating signs, beginning and ending with $+$.
   (b) Using your answers to part (a) (and more calculations if you need them), guess a simple formula for $\binom{2}{2} - \binom{3}{2} + \binom{4}{2} - \cdots - \binom{2n-1}{2} + \binom{2n}{2}$ in terms of n.
   (c) Use *induction* (or well ordering) to prove that your guess in part (b) is correct for all positive integers n.
   (d) Give a *combinatorial proof* (for example, see pages 357 and 360) that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} = \binom{n}{3}$$

   for all integers $n \geq 3$. [*Hint*: addition rule. How many 3-element subsets of $\{1, 2, \ldots, n\}$ have largest element n? How many have largest element $n - 1$? And so on.]