MATH 271 ASSIGNMENT 4 SOLUTIONS

1. If $f : \mathbb{R} \to \mathbb{R}$ is a function (where $\mathbb{R}$ is the set of all real numbers), we define the function $f^{(2)}$ to be the composition $f \circ f$, and for any integer $n \geq 2$, define $f^{(n+1)} = f \circ f^{(n)}$. So $f^{(2)}(x) = (f \circ f)(x) = f(f(x))$, $f^{(3)}(x) = (f \circ f^{(2)})(x) = f(f(f(x)))$, and so on. We also define $f^{(1)}$ to be f.

   (a) Use Theorem 3.5.1 on page 167 to prove that $\lfloor x - \lfloor x \rfloor \rfloor = 0$ for every real number x.

   (b) Suppose that $f(x) = x - \lfloor x \rfloor$ for all $x \in \mathbb{R}$. Calculate and simplify $f^{(2)}(x)$ and $f^{(3)}(x)$. [*Hint*: part (a).] Then guess a simple formula for $f^{(n)}(x)$ for all integers $n \geq 1$. Use induction (or well ordering) to prove your guess.

   (c) Suppose that $g(x) = x + \lfloor x \rfloor$ for all $x \in \mathbb{R}$. Calculate and simplify $g^{(2)}(x)$ and $g^{(3)}(x)$ (and more if you need them). [*Hint*: Theorem 3.5.1 on page 167.] Then guess a simple formula for $g^{(n)}(x)$ for all integers $n \geq 1$. Use induction (or well ordering) to prove your guess.

(a) Let x be a real number. By Theorem 3.5.1, since $-\lfloor x \rfloor \in \mathbb{Z}$,

$$\lfloor x - \lfloor x \rfloor \rfloor = \lfloor x + (-\lfloor x \rfloor) \rfloor = \lfloor x \rfloor + (-\lfloor x \rfloor) = 0.$$

(b) From part (a) we get that

$$f^{(2)}(x) = f(f(x)) = f(x - \lfloor x \rfloor) = (x - \lfloor x \rfloor) - \lfloor x - \lfloor x \rfloor \rfloor = x - \lfloor x \rfloor = f(x).$$

From this it is clear that $f^{(3)}(x) = (f \circ f^{(2)})(x) = (f \circ f)(x) = f^{(2)}(x) = f(x)$ too.
We guess that $f^{(n)}(x) = x - \lfloor x \rfloor = f(x)$ for all integers $n \geq 1$. The basis step is the case $n = 1$, which just says that $f^{(1)}(x) = f(x)$, which is true by definition above. Note that the formula is true for $n = 2$ too, as we already showed above. Now for the inductive step, suppose that $f^{(k)}(x) = f(x)$ for some integer $k \geq 1$. Then

$$\begin{aligned} f^{(k+1)}(x) &= (f \circ f^{(k)})(x) = (f \circ f)(x) && \text{by the assumption} \\ &= f^{(2)}(x) = f(x) && \text{by the case } n = 2, \end{aligned}$$

so the statement is true for the integer $n = k + 1$. Therefore the statement is true for all integers $n \geq 1$.

(c) Well,

$$\begin{aligned} g^{(2)}(x) &= g(g(x)) = g(x + \lfloor x \rfloor) \\ &= (x + \lfloor x \rfloor) + \lfloor x + \lfloor x \rfloor \rfloor \\ &= (x + \lfloor x \rfloor) + \lfloor x \rfloor + \lfloor x \rfloor && \text{by Theorem 3.5.1} \\ &= x + 3\lfloor x \rfloor \end{aligned}$$

And then

$$\begin{aligned} g^{(3)}(x) &= (g \circ g^{(2)})(x) = g(x + 3\lfloor x \rfloor) \\ &= (x + 3\lfloor x \rfloor) + \lfloor x + 3\lfloor x \rfloor \rfloor \\ &= (x + 3\lfloor x \rfloor) + \lfloor x \rfloor + 3\lfloor x \rfloor && \text{by Theorem 3.5.1} \\ &= x + 7\lfloor x \rfloor. \end{aligned}$$

And of course $g^{(1)}(x) = g(x) = x + \lfloor x \rfloor$. We guess (maybe after working out $g^{(4)}(x)$ too) that $g^{(n)}(x) = x + (2^n - 1)\lfloor x \rfloor$ for all integers $n \geq 1$. Here is a proof of this formula by induction.

The basis step is the case $n = 1$, which says $g^{(1)}(x) = x + (2^1 - 1)\lfloor x \rfloor = x + \lfloor x \rfloor$, which is true. So suppose that $g^{(k)}(x) = x + (2^k - 1)\lfloor x \rfloor$ for some integer $k \geq 1$. Then

$$\begin{aligned} g^{(k+1)}(x) &= (g \circ g^{(k)})(x) = g(x + (2^k - 1)\lfloor x \rfloor) && \text{by assumption} \\ &= (x + (2^k - 1)\lfloor x \rfloor) + \lfloor x + (2^k - 1)\lfloor x \rfloor \rfloor \\ &= (x + (2^k - 1)\lfloor x \rfloor) + \lfloor x \rfloor + (2^k - 1)\lfloor x \rfloor && \text{by Theorem 3.5.1} \\ &= x + (2^k - 1 + 1 + 2^k - 1)\lfloor x \rfloor \\ &= x + (2^{k+1} - 1)\lfloor x \rfloor. \end{aligned}$$

so the formula is true for the integer $n = k + 1$. Therefore the formula is true for all integers $n \geq 1$.

2. Let $[n] = \{1, 2, 3, \ldots, n\}$, where $n \geq 3$ is an integer.

   (a) Define the relation $\mathscr{R}$ on the power set $\mathscr{P}([n])$ by: for all sets $A, B \in \mathscr{P}([n])$, $A \mathscr{R} B$ if and only if $A - B = \{1, 2\}$. Is $\mathscr{R}$ reflexive? Symmetric? Transitive? Give reasons.

   (b) Find the number of sets $B \in \mathscr{P}([n])$ so that $\{1, 2, 3\} \mathscr{R} B$.

   (c) Define the relation $\mathscr{S}$ on the power set $\mathscr{P}([n])$ by: for all sets $A, B \in \mathscr{P}([n])$, $A \mathscr{S} B$ if and only if $A - B \subseteq \{1, 2\}$. Is $\mathscr{S}$ reflexive? Symmetric? Transitive? Give reasons.

   (a) $\mathscr{R}$ **is not reflexive**. For example, let $A = \emptyset \in \mathscr{P}([n])$. Then $A - A = \emptyset \neq \{1, 2\}$, so $A \not\mathscr{R} A$. (In fact, no set $A \in \mathscr{P}([n])$ is related to itself, for a similar reason.)

   $\mathscr{R}$ **is not symmetric**. For example, let $A = \{1, 2\}$ and $B = \emptyset$. Then $A, B \in \mathscr{P}([n])$ and $A - B = \{1, 2\}$, so $A \mathscr{R} B$, but $B - A = \emptyset \neq \{1, 2\}$, so $B \not\mathscr{R} A$.

   $\mathscr{R}$ **is transitive**. Suppose that $A, B, C \in \mathscr{P}([n])$ are such that $A \mathscr{R} B$ and $B \mathscr{R} C$. This means that $A - B = \{1, 2\}$ and $B - C = \{1, 2\}$. But $A - B = \{1, 2\}$ means in particular that $1 \notin B$, while $B - C = \{1, 2\}$ means in particular that $1 \in B$. This is a contradiction, which shows that the "if" part "$A \mathscr{R} B$ and $B \mathscr{R} C$" of the definition of transitivity cannot happen. Thus $\mathscr{R}$ is transitive vacuously.

   (b) $\{1, 2, 3\} \mathscr{R} B$ means that $\{1, 2, 3\} - B = \{1, 2\}$, which happens exactly if $1 \notin B$, $2 \notin B$, and $3 \in B$. The other $n - 3$ elements $\{4, 5, \ldots, n\}$ of $[n]$ can either be in B or not, it doesn't matter because $\{1, 2, 3\} \mathscr{R} B$ will be true regardless. Thus there are exactly 2^{n-3} such sets $B \in \mathscr{P}([n])$.

   (c) $\mathscr{S}$ **is reflexive**. Let $A \in \mathscr{P}([n])$ be arbitrary. Then $A - A = \emptyset \subseteq \{1, 2\}$, so $A \mathscr{S} A$.

   $\mathscr{S}$ **is not symmetric**. For example, let $A = \{1, 2\}$ and $B = \{3\}$. Then $A, B \in \mathscr{P}([n])$ and $A - B = \{1, 2\} \subseteq \{1, 2\}$, so $A \mathscr{S} B$, but $B - A = \{3\} \not\subseteq \{1, 2\}$, so $B \not\mathscr{S} A$.

   $\mathscr{S}$ **is transitive**. Suppose that $A, B, C \in \mathscr{P}([n])$ are such that $A \mathscr{S} B$ and $B \mathscr{S} C$. This means that $A - B \subseteq \{1, 2\}$ and $B - C \subseteq \{1, 2\}$. We want to prove that $A \mathscr{S} C$, which means we want to prove that $A - C \subseteq \{1, 2\}$. Let $a \in A - C$ be arbitrary. This means $a \in A$ and $a \notin C$. Now look at two cases.

Case (i). If $a \notin B$, then we would get $a \in A - B$ and thus $a \in \{1, 2\}$, since $A - B \subseteq \{1, 2\}$.
Case (ii). If $a \in B$, then since $a \notin C$, we get that $a \in B - C$ and thus $a \in \{1, 2\}$ since $B - C \subseteq \{1, 2\}$.
So $a \in \{1, 2\}$ in either case. Therefore $A - C \subseteq \{1, 2\}$, which completes the proof that $\mathscr{S}$ is transitive.

3. For sets A and B, define a relation R on $A \cup B$ by: for all $x, y \in A \cup B$, xRy if and only if $(x, y) \in A \times B$. Prove or disprove each of the following statements.

   (a) For all sets A and B, if R is reflexive then $A = B$.

   (b) For all sets A and B, if R is symmetric then $A = B$.

   (c) For all nonempty sets A and B, if R is symmetric then $A = B$.

   (d) For all nonempty sets A and B, if R is transitive then $A = B$.

   (e) For all sets A and B, if $A = B$ then R is an equivalence relation.

   (a) This statement is **true**. Here is a proof. Suppose that A and B are sets so that R is reflexive. We want to prove that $A = B$. It is enough to prove that $A \subseteq B$. So let $a \in A$ be arbitrary. Since R is reflexive and $a \in A \cup B$, aRa must be true, which means that $(a, a) \in A \times B$. But this means that $a \in B$. Therefore $A \subseteq B$. In the same way we could prove that $B \subseteq A$, so $A = B$.

   (b) This statement is **false**. A counterexample is $A = \{1\}$ and $B = \emptyset$. Then $A \times B = \emptyset$ (this is Exercise 27 on page 292), so no elements can be related by R. Thus $R = \emptyset$, and so R is symmetric vacuously. However $A \neq B$.

   (c) This statement is **true**. Here is a proof. Suppose that A and B are nonempty sets so that R is symmetric. We want to prove that $A = B$. Once again it is enough to prove that $A \subseteq B$. So let $a \in A$ be arbitrary, and let b be any element in B, which we know exists because $B \neq \emptyset$. Then $(a, b) \in A \times B$, so aRb. Since R is symmetric, this means that bRa must be true, so $(b, a) \in A \times B$. But this means that $a \in B$. Therefore $A \subseteq B$. In the same way we could prove that $B \subseteq A$, so $A = B$.

   (d) This statement is **false**. A counterexample is $A = \{1\}$ and $B = \emptyset$. Then again $A \times B = \emptyset$ and so $R = \emptyset$. Thus R is transitive vacuously. However $A \neq B$.

   (e) This statement is **true**. Here is a proof. Let $A = B$ be an arbitrary set. Then $A \cup B = A$, so R is defined by: for all $x, y \in A$, xRy if and only if $(x, y) \in A \times A$. But $(x, y) \in A \times A$ for all $x, y \in A$, so xRy for all $x, y \in A$. Thus it is clear that R is reflexive, symmetric and transitive, so R is an equivalence relation.

   Note that in this case R has just one equivalence class, namely all of A, since $[x] = A$ for all $x \in A$.