MATHEMATICS 271 WINTER 2010
MIDTERM SOLUTIONS

[6] **1**. Use the Euclidean algorithm to find $\gcd(105, 17)$. Then use your work to write $\gcd(105, 17)$ in the form $105a + 17b$ where a and b are integers.
Solution: We have

$$\begin{aligned} 105 &= 6 \times 17 + 3 \\ 17 &= 5 \times 3 + 2 \\ 3 &= 1 \times 2 + 1, \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

and so $\gcd(105, 17) = 1$, and
$\gcd(105, 17) = 1 = 3 - 2 = 3 - (17 - 5 \times 3) = 6 \times 3 - 17 = 6 \times (105 - 6 \times 17) - 17 = 6 \times 105 - 37 \times 17$.

Another way is to use the "table method" as follows.

		105	17
	105	1	0
	17	0	1
$R_1 - 6R_2$	3	1	-6
$R_2 - 5R_3$	2	-5	31
$R_3 - R_4$	1	6	-37

Thus, $\gcd(105, 17) = 1$ and $\gcd(105, 17) = 105 \times 6 + 17 \times (-37)$.

2. Let $X = \{1, 2, 3, 4\}$.
[3] (a) Find two subsets A and B of X so that $A - B \neq \emptyset$ but B has more elements than A does. Be sure to explain your answer.
Solution: Let $A = \{1\}$ and $B = \{2, 3\}$. Then A and B are subsets of X, and B has more elements than A does, but $A - B = \{1\} \neq \emptyset$.

[3] (b) Find an element of $X \times \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X. Explain.
Solution: An element of $X \times \mathcal{P}(X)$ is the ordered pair $(1, \emptyset)$. This is because $1 \in X$ and $\emptyset \in \mathcal{P}(X)$ (since $\emptyset \subseteq X$). Another solution is $(1, \{2\})$.

3. One of the following statements is true and one is false. Prove the true statement. Write out the *negation* of the false statement and prove that. ($\mathbb{Z}^+$ is the set of all positive integers.)
[5] (a) For all $a, b \in \mathbb{Z}^+$, if a is composite and $a \mid b$ then b is composite.
Solution: This is true and here is a proof. Let $a, b \in \mathbb{Z}^+$ and suppose that a is composite and that $a \mid b$. Since a is composite, there exist positive integers $r > 1$ and $s > 1$ such that $a = rs$. Since $a \mid b$, there is an integer k such that $b = ak$. Since $b = ak$ and a and b are positive, we see that k is also positive. Now, $b = ak = rsk$ where $r > 1$ and $sk > 1$ are integers ($sk > 1$ because $s > 1$ and $k \geq 1$). It follows that b is composite.

[4] (b) For all $a, b \in \mathbb{Z}^+$, if a is prime and $b \mid a$ then b is prime.
Solution: This statement is false. Its negation is "There exist $a, b \in \mathbb{Z}^+$ such that a is prime and $b \mid a$, but b is not prime."

For example, let $a = 2$ and $b = 1$. Then $a, b \in \mathbb{Z}^+$, a is prime and $b \mid a$ (because $a = b \times 2$ and $2 \in \mathbb{Z}$). However, $b = 1$ is not prime.

4. Let $\mathcal{S}$ be the statement:

$$\text{for all sets } A,\ B \text{ and } C, \text{ if } A \cap B = \emptyset \text{ and } C \subseteq B \text{ then } A \cap C = \emptyset.$$

[4] (a) Prove $\mathcal{S}$, using contradiction and the element method.
Solution: Let A, B and C be sets such that $A \cap B = \emptyset$ and $C \subseteq B$. We prove that $A \cap C = \emptyset$ by contradiction. Suppose that $A \cap C \neq \emptyset$, that is, there exists an element $x \in A \cap C$. Since $x \in A \cap C$, we know $x \in A$ and $x \in C$. Since $x \in C$ and $C \subseteq B$, we get $x \in B$. Since $x \in A$ and $x \in B$, we get $x \in A \cap B$. Thus, there exists $x \in A \cap B$ and hence, $A \cap B \neq \emptyset$ which contradicts the assumption that $A \cap B = \emptyset$. Therefore, $A \cap C = \emptyset$.

[4] (b) Write out (as simply as possible) the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.
Solution: This converse of $\mathcal{S}$ is: "For all sets A, B and C, if $A \cap C = \emptyset$ then $A \cap B = \emptyset$ and $C \subseteq B$."

The converse of $\mathcal{S}$ is false. We are to prove its negation which is: " There exist sets A, B and C such that $A \cap C = \emptyset$, but $A \cap B \neq \emptyset$ or $C \not\subseteq B$." For example, let $A = B = \emptyset$ and $C = \{1\}$. Then $A \cap C = \emptyset \cap C = \emptyset$, and $C \not\subseteq B$ because $1 \in C$ but $1 \notin B$.

[3] (c) Write out (as simply as possible) the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.
Solution: This contrapositive of $\mathcal{S}$ is: "For all sets A, B and C, if $A \cap C \neq \emptyset$ then $A \cap B \neq \emptyset$ or $C \not\subseteq B$."

The contrapositive of $\mathcal{S}$ is true because it is logically equivalent to $\mathcal{S}$ which is true by part (a).

[8] **6**. Prove **using mathematical induction** (or well ordering) that

$$\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + ... + \frac{n}{n+2} \geq \frac{n}{3}$$

for all integers $n \geq 1$.
Solution: Note that $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + ... + \frac{n}{n+2} = \sum_{i=1}^{n} \frac{i}{i+2}$. We now prove that $\sum_{i=1}^{n} \frac{i}{i+2} \geq \frac{n}{3}$ for all integers $n \geq 1$ using mathematical induction on n.
Basis step: $(n = 1)$

We have $\sum_{i=1}^{1} \frac{i}{i+2} = \frac{1}{1+2} = \frac{1}{3} \geq \frac{1}{3}$. Thus, the statement is true for the case $n = 1$.

Inductive step: Let $k \geq 1$ be an integer and suppose that $\sum_{i=1}^{k} \frac{i}{i+2} \geq \frac{k}{3}$. We want to show that $\sum_{i=1}^{k+1} \frac{i}{i+2} \geq \frac{k+1}{3}$.

Now,

$$
\begin{aligned}
\sum_{i=1}^{k+1} \frac{i}{i+2} &= \sum_{i=1}^{k} \frac{i}{i+2} + \frac{k+1}{k+3} \\
&\geq \frac{k}{3} + \frac{k+1}{k+3} && \text{by the inductive hypothesis} \\
&= \frac{k(k+3) + 3(k+1)}{3(k+3)} \\
&= \frac{k^2 + 3k + 3k + 3}{3(k+3)} \\
&= \frac{k^2 + 6k + 3}{3(k+3)} \\
&> \frac{k^2 + 4k + 3}{3(k+3)} && \text{because } 6k > 4k \text{ when } k \geq 1 \\
&= \frac{(k+1)(k+3)}{3(k+3)} \\
&= \frac{(k+1)}{3}
\end{aligned}
$$

Thus, we proved the inductive step.

Therefore, by the Principle of Mathematical Induction, we conclude that

$$\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + ... + \frac{n}{n+2} \geq \frac{n}{3} \text{ for all integers } n \geq 1.$$