

Math 381 KEY
Assignment II

12.1 Problems 24

$$f(x, y) = \frac{y}{x^2 + y^2} \quad (x, y) \neq (0, 0)$$

level curve $f(x, y) = 0 \Rightarrow y = 0$ - Punctured

x -axis

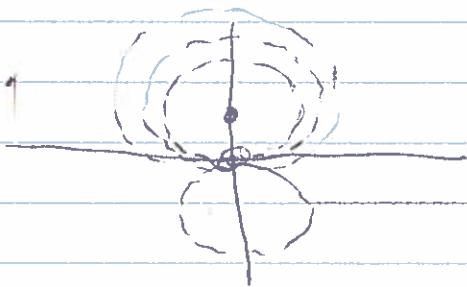
level curve $f(x, y) = k \neq 0$ Set $\frac{1}{k} = l$

Then $\frac{y}{x^2 + y^2} = \frac{1}{l}$ and $ly = x^2 + y^2$

whence $x^2 + y^2 - ly = 0$ so $x^2 + (y - \frac{l}{2})^2 = \frac{l^2}{4}$

which is a circle of radius $\frac{|l|}{2} = \frac{1}{|2k|}$ and center

at $(0, \frac{l}{2})$.



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Problem 36 The level surfaces of $f(x, y, z) = C$ is

a plane having intercepts $\frac{C}{6}, \frac{2C}{3}, \frac{3C}{3}$ on the

x -axis, y -axis and z -axis respectively. An equation

of the plane is $6x + 3y + 2z = C$, so $f(x, y, z) = \left(\frac{6x + 3y + 2z}{6}\right)^{1/3}$

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§12.2 Problem 12 Consider $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{2x^4 + y^4}$

Along the coordinate axes

$$y=0 \quad \lim_{x \rightarrow 0} \frac{x^2 \cdot 0}{2x^4 + 0} = 0$$

$$\text{and } x=0 \quad \lim_{y \rightarrow 0} \frac{0 \cdot y^2}{2 \cdot 0 + y^4} = 0$$

Along the line $y=kx$, $k \neq 0$, $x \neq 0$

$$\lim_{x \rightarrow 0} \frac{x^2 (kx)^2}{2x^4 + (kx)^4} = \lim_{x \rightarrow 0} \frac{k^2}{2+k^4} \neq 0,$$

so $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{2x^4 + y^4}$ does not exist.

Problem 14 Consider $f(x,y) = \frac{x^3 - y^3}{x - y}$ $x \neq y$.

$$\text{Now } x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

$$\text{so for } x \neq y \quad f(x,y) = x^2 + xy + y^2$$

If we define $f(x,y) = x^2 + xy + y^2$ when $x = y$,

then f is continuous on \mathbb{R}^2 .

§12.3 Problem 12 Consider $f(x,y) = \begin{cases} \frac{x^2-2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x=y \end{cases}$

For $D_1 f(0,0)$ consider $\lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{h^2 - 2 \cdot 0^2 - 0}{h} = 1$$

For $D_2 f(0,0)$ consider $\lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k}$

$$= \lim_{k \rightarrow 0} \frac{0^2 - 2k^2 - 0}{0 - k} = -1$$

Problem 16 Let $f(x,y) = e^{xy}$

Then $f_1(x,y) = ye^{xy}$ and $f_2(x,y) = xe^{xy}$

At $(2,0)$

$$\begin{aligned} f(2,0) &= e^0 = 1 \\ f_1(2,0) &= 0e^0 = 0 \\ f_2(2,0) &= 2e^0 = 2 \end{aligned}$$

So a normal to the tangent plane is $(0, 2, -1)$

An equation of the normal line is $(x,y,z) = (2,0,1) + t(0,2,-1)$

and an equation of the tangent plane is $z = 1 + 0(x-2) + 2(y-0)$
or $z = 1 + 2y$

Problem 24. Consider the surface with equation

$$z = xy e^{-\frac{(x^2+y^2)}{2}}$$

Then $\frac{\partial z}{\partial x} = y e^{-\frac{(x^2+y^2)}{2}} - x y^2 e^{-\frac{(x^2+y^2)}{2}}$

and $\frac{\partial z}{\partial y} = x e^{-\frac{(x^2+y^2)}{2}} - x^2 y e^{-\frac{(x^2+y^2)}{2}}$

A normal direction to the tangent plane at (x, y, z) is $(f_1, f_2, -1)$ or $(y e^{-\frac{(x^2+y^2)}{2}} (1-2x^2), x e^{-\frac{(x^2+y^2)}{2}} (1-2y^2), -1)$

For the tangent plane to be horizontal the normal must be parallel to $(0, 0, 1)$ so we get

$$y e^{-\frac{(x^2+y^2)}{2}} (1-2x^2) = 0$$

$$x e^{-\frac{(x^2+y^2)}{2}} (1-2y^2) = 0$$

Since $e^{-\frac{(x^2+y^2)}{2}} \neq 0$ we see

that $y(1-2x^2) = 0$

and $x(1-2y^2) = 0$

This gives $(x, y) = (0, 0)$, $(x, y) = (\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})$

Problem 26, Let $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Along $x=0$ $\lim_{y \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \frac{2 \cdot 0 \cdot y}{0^2 + y^2} = 0$

$= \lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \frac{2 \cdot x \cdot 0}{x^2 + 0^2} = 0$

But along $y=kx$ $k \neq 0$

$$\begin{aligned} \lim_{x \rightarrow 0} f(x, kx) &= \lim_{x \rightarrow 0} \frac{2 \cdot x \cdot kx}{x^2 + k^2 x^2} \\ &= \lim_{x \rightarrow 0} \frac{2k}{1+k^2} \neq 0 \end{aligned}$$

so f is not continuous at $(0, 0)$.

To calculate $f_1(0, 0)$ consider

$$\lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{2 \cdot h \cdot 0}{h^2 + 0^2} = 0$$

$$\text{and } \lim_{k \rightarrow 0} \frac{f(0, 0+k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{2 \cdot 0 \cdot k}{0^2 + k^2} = 0$$

Shows $f_1(0, 0) = 0 = f_2(0, 0)$.

§ 12.4 Problem 4
Let $f(x, y) = z = \sqrt{3x^2 + y^2}$

$$\text{Then } \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{3x^2 + y^2}} \cdot 6x = \frac{3x}{\sqrt{3x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{3x^2 + y^2}} \cdot 2y = \frac{y}{\sqrt{3x^2 + y^2}}$$

$$\begin{aligned} \text{So } \frac{\partial^2 z}{\partial x^2} &= \frac{3}{\sqrt{3x^2 + y^2}} + 3x \left(\frac{-1}{2} \right) \cdot \frac{6x}{(3x^2 + y^2)^{3/2}} \\ &= \frac{3(3x^2 + y^2) - 9x^2}{(3x^2 + y^2)^{3/2}} = \frac{3y^2}{(3x^2 + y^2)^{3/2}} \end{aligned}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-1}{2} \cdot \frac{3x}{(3x^2 + y^2)^{3/2}} \cdot 2y = \frac{-3xy}{(3x^2 + y^2)^{3/2}}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= \frac{1}{\sqrt{3x^2 + y^2}} - \frac{1}{2} \cdot \frac{y}{(3x^2 + y^2)^{3/2}} \cdot 2y \\ &= \frac{3x^2 + y^2 - y^2}{(3x^2 + y^2)^{3/2}} = \frac{3x^2}{(3x^2 + y^2)^{3/2}} \end{aligned}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{2} \cdot \frac{y}{(3x^2 + y^2)^{3/2}} \cdot 6x = \frac{-3xy}{(3x^2 + y^2)^{3/2}}$$

$$\text{so } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Problem 10 Let $z = f(x, y) = \frac{x}{x^2 + y^2}$, $(x, y) \neq (0, 0)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + y^2} - \frac{x}{(x^2 + y^2)^2} \cdot 2x$$

$$= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-2x}{(x^2 + y^2)^2} + (y^2 - x^2)(-2) \frac{2x}{(x^2 + y^2)^3}$$

$$= \frac{-2x(x^2 + y^2) - 4x(y^2 - x^2)}{(x^2 + y^2)^3}$$

$$= \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}$$

$$\frac{\partial z}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2x}{(x^2 + y^2)^2} - 2xy(-2) \frac{2y}{(x^2 + y^2)^3}$$

$$= \frac{-2x(x^2 + y^2) + 8xy^2}{(x^2 + y^2)^3} = \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3} + \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3} = 0$$

for $(x, y) \neq (0, 0)$, as required.

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Problem 16

$$\text{Let } F(x,y) = \begin{cases} \frac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$

$$F_1(x,y) = \frac{2y(x^2-y^2) + 4x^2y}{(x^2+y^2)} - \frac{2xy(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$= \frac{(6x^2y - 2y^3)(x^2+y^2) - 4x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{6x^4y + 6x^2y^3 - 2x^2y^3 - 2y^5 - 4x^4y + 4x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{2x^4y + 8x^2y^3 - 2y^5}{(x^2+y^2)^2}$$

$$F_2(x,y) = \frac{2xc(x^2-y^2) - 4xy^2}{x^2+y^2} - \frac{2xy(x^2-y^2)2y}{(x^2+y^2)^2}$$

$$= \frac{(2x^3 - 6xy^2)(x^2+y^2) - 4xy^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{2x^5 + 2x^3y^2 - 6x^3y^2 - 6xy^4 - 4x^3y^2 + 4xy^4}{(x^2+y^2)^2}$$

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$$= \frac{2x^5 - 8x^3y^2 - 2xy^4}{(x^2+y^2)^2}$$

$$F_{11}(x, y) = \frac{8x^3y + 16xy^3}{(x^2+y^2)^2} - \frac{2(2x^4y + 8x^2y^3 - 2y^5)2x}{(x^2+y^2)^3}$$

$$= \frac{(8x^3y + 16xy^3)(x^2+y^2) - 8x^5y - 32x^3y^3 + 8xy^5}{(x^2+y^2)^3}$$

$$= \frac{8x^5y + 8x^3y^3 + 16x^3y^3 + 16xy^5 - 8x^5y - 32x^3y^3 + 8xy^5}{(x^2+y^2)^3}$$

$$= \frac{-8x^3y^3 + 24xy^5}{(x^2+y^2)^3}$$

$$F_{12}(x, y) = \frac{2x^4 + 24x^2y^2 - 10y^4}{(x^2+y^2)^2} - \frac{2(2x^4y + 8x^2y^3 - 2y^5)2y}{(x^2+y^2)^3}$$

$$= \frac{(2x^4 + 24x^2y^2 - 10y^4)(x^2+y^2) - 8x^4y^2 - 32x^2y^4 + 8y^6}{(x^2+y^2)^3}$$

$$= \frac{2x^6 + 24x^4y^2 - 10x^2y^4 + 2x^4y^2 + 24x^2y^4 - 10y^6 - 8x^4y^2 - 32x^2y^4 + 8y^6}{(x^2+y^2)^3}$$

$$= \frac{2x^6 + 18x^4y^2 - 18x^2y^4 + 8y^6}{(x^2+y^2)^3}$$

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$$\begin{aligned}F_{22}(x, y) &= \frac{-16x^3y - 8xy^3}{(x^2 + y^2)^2} - \frac{2(2x^5 - 8x^3y^2 - 2xy^4)(2y)}{(x^2 + y^2)^3} \\&= \frac{(-16x^3y - 8xy^3)(x^2 + y^2) - 8x^5y + 32x^3y^3 + 8xy^5}{(x^2 + y^2)^3} \\&= \frac{-16x^5y - 16x^3y^3 - 8x^3y^3 - 8xy^5 - 8x^5y + 32x^3y^3 + 8xy^5}{(x^2 + y^2)^3} \\&= \frac{-24x^5y + 8x^3y^3}{(x^2 + y^2)^3}\end{aligned}$$

$$\begin{aligned}F_{21}(x, y) &= \frac{10x^4 - 24x^2y^2 - 2y^4}{(x^2 + y^2)^2} - \frac{2(2x^5 - 8x^3y^2 - 2xy^4)2x}{(x^2 + y^2)^3} \\&= \frac{(10x^4 - 24x^2y^2 - 2y^4)(x^2 + y^2) - 8x^6 + 32x^4y^2 + 8x^2y^4}{(x^2 + y^2)^3} \\&= \frac{10x^6 + 10x^4y^2 - 24x^4y^2 - 24x^2y^4 - 2x^2y^4 - 2y^6 - 8x^6 + 32x^4y^2 + 8x^2y^4}{(x^2 + y^2)^3} \\&= \frac{2x^6 + 18x^4y^2 - 18x^2y^4 - 2y^6}{(x^2 + y^2)^3}\end{aligned}$$

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At $(0,0)$

$$F_1(0,0) = \lim_{h \rightarrow 0} \frac{F(0+h, 0) - F(0,0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

and $F_2(0,0) = \lim_{k \rightarrow 0} \frac{F(0, 0+k) - F(0,0)}{k}$

$$= \lim_{k \rightarrow 0} \frac{0-0}{k} = 0$$

$$F_{12}(0,0) = \lim_{k \rightarrow 0} \frac{F_1(0, 0+k) - F_1(0,0)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{-2k^5 \text{ or } \text{---} - 0}{(0+k^2)^2} = -2$$

$$= \lim_{k \rightarrow 0} \frac{-2k^5}{k^4} = -2$$

$$F_{21}(0,0) = \lim_{h \rightarrow 0} \frac{F_2(0+h, 0) - F_2(0,0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h^5}{(h^2+0)^2} - 0 = \lim_{h \rightarrow 0} \frac{2h^5}{h^4} = 2$$

The conditions to check are that $F_1(x,y)$ $F_2(x,y)$ are continuous

in a neighborhood of $(0,0)$ and that F_2 and F_3 are continuous at $(0,0)$.

F_1 and F_2 as well as F_{12} , F_{21} are continuous when $(x,y) \neq (0,0)$ so the problem is to analyze their behaviour at $(0,0)$.

$$\text{Consider } F_1(x,y) = y \frac{2x^4 + 8x^2y^2 - 2y^4}{(x^2+y^2)^2}$$

$$\text{Now } \left| \frac{2x^4 + 8x^2y^2 - 2y^4}{(x^2+y^2)^2} \right| \leq 4$$

$$\text{So } \lim_{(x,y) \rightarrow (0,0)} F_1(x,y) = 0 = F_1(0,0)$$

$$\text{Also } F_2(x,y) = x \frac{2x^4 - 8x^2y^2 - 2y^4}{(x^2+y^2)^2}$$

$$\text{and similarly } \lim_{(x,y) \rightarrow (0,0)} F_2(x,y) = 0.$$

$$\text{However } \lim_{k \rightarrow 0} F_{12}(0,k) = \lim_{k \rightarrow 0} \frac{-2k^6}{(0^2+k^2)^3} = -2$$

$$\text{while } \lim_{h \rightarrow 0} F_{12}(h,0) = \lim_{h \rightarrow 0} \frac{2h^6}{(h^2+0^2)^3} = 2 \text{ so } F_{12}$$

is not continuous at $(0,0)$, so there is no contradiction.

§ 12.5 Problem 16

Suppose that f is a harmonic function of two variables, that is $f_{11} + f_{22} = 0$. Let $u = \frac{x}{x^2 + y^2}$, $v = \frac{-y}{x^2 + y^2}$

Note that u is harmonic (except at $(0,0)$)

$$\text{Since } \frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}, \quad \frac{\partial^2 v}{\partial y^2} = \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3}$$

Similarly v is harmonic, that is $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Now let $w = f(u, v)$

$$\text{Then } \frac{\partial w}{\partial x} = f_1(u, v) \frac{\partial u}{\partial x} + f_2(u, v) \frac{\partial v}{\partial x}$$

$$\begin{aligned} \text{So } \frac{\partial^2 w}{\partial x^2} &= f_{11}(u, v) \left(\frac{\partial u}{\partial x} \right)^2 + f_{21}(u, v) \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + f_1(u, v) \frac{\partial^2 u}{\partial x^2} \\ &\quad + f_{12}(u, v) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + f_{22}(u, v) \left(\frac{\partial v}{\partial x} \right)^2 + f_2(u, v) \frac{\partial^2 v}{\partial x^2} \end{aligned}$$

$$\text{Now } \frac{\partial w}{\partial y} = f_1(u, v) \frac{\partial u}{\partial y} + f_2(u, v) \frac{\partial v}{\partial y}$$

$$\text{So } \frac{\partial^2 w}{\partial y^2} = f_{11}(u, v) \left(\frac{\partial u}{\partial y} \right)^2 + f_{21}(u, v) \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + f_1(u, v) \frac{\partial^2 u}{\partial y^2} \\ + f_{12}(u, v) \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + f_{22}(u, v) \left(\frac{\partial v}{\partial y} \right)^2 + f_2(u, v) \frac{\partial^2 v}{\partial y^2}$$

Now $f_{12}(u, v) = f_{21}(u, v)$

$$\text{So } \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f_{11}(u, v) \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) \\ + f_{22}(u, v) \left(\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right) \\ + 2 f_{12}(u, v) \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] \\ + f_1(u, v) \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + f_2(u, v) \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$$

Note that $\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left(\frac{y^2 - x^2}{(x^2 + y^2)^2} \right)^2 + \left(\frac{-2xy}{(x^2 + y^2)^2} \right)^2$

and $\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 = \left(\frac{2xy}{(x^2 + y^2)^2} \right)^2 + \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right)^2$

$$\text{So } \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = (f_{11}(u, v) + f_{22}(u, v)) \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) \\ + 2 f_{12}(u, v) \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] \\ + f_1(u, v) \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] + f_2(u, v) \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$$

which, using that f, u, v are harmonic reduces to
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 2 f_{12}(u, v) \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right]$$

But
$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \frac{2xy}{(x^2 + y^2)^2} + \left(\frac{-2xy}{(x^2 + y^2)^2} \right) \left(\frac{y^2 - x^2}{(x^2 + y^2)^2} \right)$$
$$= 0$$

So $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$ and w is harmonic.

Problem 30 From 29b $F_1(x, y) = -F_2(y, x)$

and $F_{12}(x, y) = -F_{21}(y, x)$ for $(x, y) \neq (0, 0)$

where $F(x, y)$ is the potential of §12.4 Problem 16, discussed above.

Since $F_{12}(x, y) = -F_{21}(y, x)$ for $(x, y) \neq (0, 0)$

we get $F_{12}(x, y) = -F_{12}(y, x)$. Setting $x = y \neq 0$

we need $F_{12}(x, x) = -F_{12}(x, x)$, so $2F_{12}(x, x) = 0$

and $F_{12}(x, x) = 0$ for $x \neq 0$.

Problem 18

$$f(\rho, \phi, \theta) = (x, y, z) \text{ where}$$

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\text{Jacobian} = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \\ \rho \cos \phi \cos \theta & \rho \cos \phi \sin \theta & -\rho \sin \phi \\ -\rho \sin \phi \sin \theta & +\rho \sin \phi \cos \theta & 0 \end{bmatrix}$$

$$\text{Det(Jacobian)} = \sin \phi \cos \theta (\rho^2 \sin^2 \phi \cos \theta) - \sin \phi \sin \theta (-\rho^2 \sin^2 \phi \sin \theta)$$

$$+ \cos \phi (\rho^2 \cos \phi \sin \phi \cos^2 \theta + \rho^2 \cos \phi \sin \phi \sin^2 \theta)$$

$$= \rho^2 \sin \phi (\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta) + \rho^2 \cos^2 \phi \sin \phi (\cos^2 \theta + \sin^2 \theta)$$

$$= \rho^2 \sin \phi (\sin^2 \phi + \cos^2 \phi) = \rho^2 \sin \phi$$

From the earlier exercise we calculated that

$$F_{12}(0,0) = -2 \neq \lim_{x \rightarrow 0} F_{12}(x,x), \text{ so } F_{12} \text{ is}$$

not continuous at $(0,0)$.

§ 12.6 Problem 12 (Problem 8 in 6th Edition)

The radius and height of a right circular ^{conical} tank

are measured to be 25 ft and 21 ft respectively, each

accurate to $\frac{1}{24}$ ft.

$$V = \frac{1}{3} \pi r^2 h$$

$$\text{So } dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= \frac{2}{3} \pi r h dr + \frac{1}{3} \pi r^2 dh$$

$$dV \approx \frac{2}{3} \pi (25)(21) \left(\frac{1}{24} \right) + \frac{1}{3} \pi (25)^2 \left(\frac{1}{24} \right)$$

$$= \frac{25\pi}{3} \left(\frac{42 + 25}{24} \right) = 25\pi \frac{23}{24}$$

$$\approx \underline{75.3 \text{ ft}^3}$$

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