Term Test II

Key

Suppose $\nabla f(x,y) = (0,0)$ throughout $x^2+y^2<r^2$ and suppose $x_0^2+y_0^2<r^2$ and $x_1^2+y_1^2<r^2$ where $f(x_0,y_0) \neq f(x_1,y_1)$. Consider $h(t) = f(t(x_1,y_1)+(1-t)(x_0,y_0))$
$= f(x_0+t(x_1-x_0), y_0+t(y_1-y_0))$, on the line segment joining (x_0,y_0) to (x_1,y_1). The entire segment lies within the disk and we have $h(0) = f(x_0,y_0)$
$\neq f_1(x_1,y_1) = h(1)$. By the Chain rule h is differentiable on $(0,1)$ and continuous on $[0,1]$ so $h(1)-h(0) = h'(t)\cdot(1-0)$ for some $0<t<1$. But $h'(t) = D_{\underline{u}}f(t) = \nabla f(x_0+t(x_1-x_0), y_0+t(y_1-y_0)) \circ \underline{u} = \underline{0}$ where $\underline{u}$ is the unit vector $\frac{(x_1-x_0,\, y_1-y_0)}{a}$ where $a = \sqrt{(x_1-x_0)^2+(y-y_0)^2}$. This is a contradiction since $h(1) \neq h(0)$.

2. $F(x,y,z,u,v) = xe^y + uz - \cos v - 2$

$G(x,y,z,u,v) = u\cos y + x^2 v - yz^2 - 1$

$$\frac{\partial(F,G)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial F}{\partial u}, & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u}, & \frac{\partial G}{\partial v} \end{bmatrix}$$

$$= \det\begin{bmatrix} z & \sin v \\ \cos y & x^2 \end{bmatrix} = x^2 z - \cos y \sin v$$

At $(2,0,1,1,0)$ $\quad \frac{\partial(F,G)}{\partial(u,v)} = 4\cdot 1 - \cos 0 \sin 0 = 4 \neq$

so that (u,v) is a function of (x,y,z) in a neighborhood

Moreover $\left(\frac{\partial u}{\partial z}\right)_{x,y} = \dfrac{-\det\begin{bmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial v} \end{bmatrix}}{\frac{\partial(F,G)}{\partial(u,v)}} = \dfrac{-\det\begin{bmatrix} u & \sin v \\ -2yz & x^2 \end{bmatrix}}{x^2 z - \cos y \sin v}$

$$= -\frac{(ux^2 + 2yz\sin v)}{x^2 z - \cos y \sin v}$$

At $(2,0,1,1,0)$ this equals $-\dfrac{(1\cdot 4 + 2\cdot 0\cdot 1\cdot \sin 0)}{4\cdot 1 - \cos 0 \sin 0}$

$$= -\frac{4}{4} = -1$$

$$f(x,y,z) = x^2+y^2+z^2$$

For

$$G(x,y,z) = x^2+y^2-z^2 = 0$$
$$H(x,y,z) = x - 2z - 3 = 0$$

$$\nabla f = (2x,\ 2y,\ 2z)$$
$$\nabla G = (2x, 2y\ -2z)$$
$$\nabla H = (1,\ 0,\ -2)$$

Expressing $\nabla f = \lambda \nabla G + \mu \nabla H$

$$2x = 2\lambda x + \mu \qquad (1)$$
$$2y = 2\lambda y + 0 \qquad (2)$$
$$2z = -2\lambda z - 2\mu \qquad (3)$$

$$x^2+y^2 = z^2 \qquad (4)$$
and $$x - 2z = 3 \qquad (5)$$

$$2x(1-\lambda) = \mu \qquad (6)$$
$$2y(1-\lambda) = 0 \qquad (7)$$
$$z(1+\lambda) = -\mu \qquad (8)$$

$$x^2+y^2 = z^2 \qquad (9)$$
$$x - 2z = 3 \qquad (10)$$

From 7 $y=0$ or $\lambda = 1$

Now if $\lambda=1$, from 6 $\mu = 0$ and from 8 $z=0$

Now (9) implies $x=y=z=0$, contradicting $x-2z=3$.

So $\lambda \neq 1$ and $y=0$.

$$2x(1-\lambda) = -(1+\lambda)z \quad \text{from 6 and 8}$$
$$x^2 = z^2 \quad \text{from 9 and } y=0$$
$$x = z \text{ gives} \quad x - 2x = 3$$

So $x = -3$, $y = 0$, $z = -3$ $\quad \lambda = 3$, $\mu = 12$

is a critical point and $f(-3,0,-3) = 9+0+9 = 18$

For $x = -z$ we get $x + 2x = 3$, $x = \frac{3}{3} = 1$

$y = 0$, $z = -1$, $\quad z(1-\lambda) = +(1+\lambda) \quad \mu = \frac{4}{3}$
$$2 - 2\lambda = +1 + \lambda$$
$$1/3 = \lambda$$

and $f(1, 0, -1) = 1 + 1 = 2$

So the max is 18 and the min is 2.

4. From the Taylor formula for degree 1 with estimated error term with $k = 0$

$$f(x+h, t) = f(x,t) + f_1(x,t)h + f_2(x,t)\cdot 0 + \frac{1}{2}\Big(f_{11}(x+\theta h, t)h^2 + 2f_{12}(x+\theta h, t)h\cdot 0 + f_{22}(x+\theta h, t)\cdot 0^2\Big)$$

So $f(x+h,t) - f(x,t) - f_1(x,t)h = \frac{1}{2} f_{11}(x+\theta h, t)h^2$

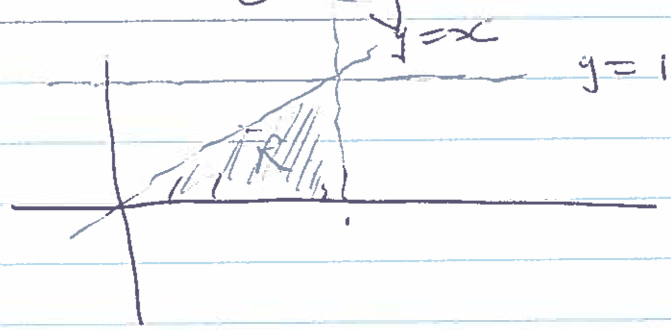
and $\left|f(x+h,t) - f(x,t) - f_1(x,t)h\right| \le \left|\frac{1}{2} f_{11}(x+\theta h, t)h^2\right| \le \frac{1}{2}g(t)\cdot$

Hence
$$\left| \frac{F(x+h)-F(x)}{h} - \int_a^b f_1(x,t)\,dt \right| \le \frac{|h|}{2}\int_a^b g(t)\,dt \le \frac{|h|}{2} K$$

which tends to 0 as $h \to 0$.

So $F'(x) = \int_a^b f_1(x,t)\,dt$ as desired.

5. I $$\int_0^1 \left(\int_y^1 e^{-x^2}\,dx \right) dy = \iint_R e^{-x^2}\,dA$$



$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{-x^2}\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left(y e^{-x^2} \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_{x=0}^{x=1} \left(x e^{-x^2} - 0 \right) dx$$

$$= -\frac{e^{-x^2}}{2} \Bigg|_0^1 = -\frac{e^{-1}}{2} + \frac{e^0}{2}$$

$$= \frac{1}{2}\left(1 - \frac{1}{e}\right).$$