Abstract

Arrangements of convex bodies are ubiquitous in discrete geometry. Covering and packing problems are two major classes of problems arising from these arrangements. This thesis makes contributions to some open questions on covering a convex body by its homothetic copies and packings of translates of a convex body. The packings we study lead to the notion of a contact graph, a tool used by mathematicians, material scientists and physicists while investigating these packings. This brings a combinatorial flavour to this thesis and leads to the algorithmic question of generating graphs and hypergraphs with desired properties.

We introduce the covering index of a convex body as a measure of how economically the body can be covered by a relatively few smaller positive homothetic copies. Determining the covering index can be thought of as a quantitative version of the sixty-year old Covering Conjecture in discrete and convex geometry. We prove several interesting properties of the covering index, compute its values for large classes of convex bodies and use it to devise an approach to attack the Covering Conjecture. We then consider the problem of determining the separable Hadwiger number of a smooth convex domain and the maximum contact numbers of totally separable translative packings of a smooth strictly convex domain. Both these questions have links with crystallography and self-assembly of material particles, and are completely solved here. Moreover, we show that for smooth Radon domains the extremal packings always lie on special kinds of lattices and characterize the underlying contact graphs. Next, we fully address the algorithmic questions of constructing tournaments from their imbalance sets and hypertournaments from their score sequences. The former relates to the Equal-Sum Subsets problem from computer science, while the latter is a special instance of the degree sequence problem that asks for the generation of all graphs or hypergraphs in some large class that have the same degree sequence. We characterize imbalance sets of tournaments, give an algorithm to generate tournaments from their imbalance sets and define a rapidly mixing Markov chain that uniformly samples all hypertournaments with a given score sequence.
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Chapter 1

Introduction

1.1 Some background

We denote the $d$-dimensional Euclidean space by $\mathbb{E}^d$ and write $\mathbb{R}^d$ instead when the norm is arbitrary or unknown. A $d$-dimensional convex body $K$ is a compact convex subset of $\mathbb{E}^d$ with nonempty interior $\text{int} K$. If $d = 2$, then $K$ is said to be a convex domain. If $K = -K$, where $-K = \{-x : x \in K\}$, then $K$ is said to be o-symmetric. Furthermore, $K$ is centrally symmetric if some translate of $K$ is o-symmetric. Since the quantities we deal with here are affine invariants, we can mostly use the two terms interchangeably. Every o-symmetric convex body $K$ in $\mathbb{E}^d$ induces a norm on $\mathbb{R}^d$, whose closed unit ball is $K$, given by

$$\|x\|_K = \inf \{\lambda > 0 : x \in \lambda K\},$$

for every $x \in \mathbb{R}^d$. We denote the corresponding normed space by $(\mathbb{R}^d, \|\cdot\|_K)$. Let $B^d$ be a $d$-dimensional Euclidean ball, $C^d$ a $d$-dimensional cube, $\Delta^d$ a $d$-simplex, and $\ell$ a line segment (or more precisely, an affine image of any of these convex bodies). If $K = B^d$ then we denote the norm $\|\cdot\|_{B^d}$ simply by $\|\cdot\|$. Thus, $\mathbb{E}^d = (\mathbb{R}^d, \|\cdot\|)$. A $d$-dimensional convex body $K$ is said to be smooth if at every point on the boundary $\partial K$ of $K$, the body $K$ is supported by a unique hyperplane of $\mathbb{E}^d$, and strictly convex if the boundary of $K$ contains no nontrivial line
segment. Given \(d\)-dimensional convex bodies \(K\) and \(L\), their Minkowski sum (vector sum) is denoted by \(K + L\) and defined as

\[
K + L = \{x + y : x \in K, y \in L\},
\]

which is a convex body in \(\mathbb{E}^d\).

A homothetic copy, or simply a homothet, of \(K\) is a set of the form \(M = \lambda K + x\), where \(\lambda\) is a nonzero real number and \(x \in \mathbb{E}^d\). If \(\lambda > 0\), then \(M\) is said to be a positive homothet and if in addition, \(\lambda < 1\), we have a smaller positive homothet of \(K\). We use the symbol \(\mathcal{K}^d\) for the metric space of \(d\)-dimensional convex bodies under the (multiplicative) Banach-Mazur distance \(d_{BM}(\cdot, \cdot)\). That is, for any \(K, L \in \mathcal{K}^d\),

\[
d_{BM}(K, L) = \inf \{\delta \geq 1 : L - b \subseteq T(K - a) \subseteq \delta(L - b), a \in K, b \in L\},
\]

where the infimum is taken over all invertible linear operators \(T : \mathbb{E}^d \rightarrow \mathbb{E}^d\) [153, Page 589]. The Banach–Mazur distance \(d_{BM}\) provides a multiplicative metric\(^1\) on \(\mathcal{K}^d\) and is used to study the continuity properties of affine invariant functionals on \(\mathcal{K}^d\).

We adopt the following notations. For \(x \neq y \in \mathbb{E}^d\), we denote the closed (resp. open) line segment in \(\mathbb{E}^d\) with end points \(x\) and \(y\) by \([x, y]\) (resp. \((x, y))\). Also given a centrally symmetric convex domain \(K\) and \(x \neq y \in \text{bd} K\), we denote the smaller (in the norm \(\|\cdot\|_K\)) of the two closed (resp. open) arcs on \(\text{bd} K\) with end points \(x\) and \(y\) by \([x, y]_K\) (resp. \((x, y)_K\)). Ties are broken arbitrarily. All arcs considered in this thesis are non-trivial, that is, different from a point.

The famous Covering Conjecture [79, 83, 122] in convex geometry – also called the Hadwiger Covering Conjecture, Levi-Hadwiger Conjecture or the Gohberg-Markus-Hadwiger Conjecture – states that any \(K \in \mathcal{K}^d\) can be covered by \(2^d\) of its smaller positive homothetic

\(^1\)One can turn the Banach–Mazur distance into an additive metric by applying \(\ln(\cdot)\). However, we make no attempt to do that.
copies, with \( 2^d \) homothets needed only if \( K \) is an affine \( d \)-cube. This conjecture appears in several equivalent forms, one of which we discuss here. Boltyanski [40] and Hadwiger [85] introduced two notions of illumination of a convex body, the former being ‘illumination by directions’ and the latter being ‘illumination by points’. The two notions are actually equivalent [40], and \( K \) is said to be illuminated if all points on the boundary of \( K \) are illuminated (in either sense). The illumination number \( I(K) \) of \( K \) is the smallest \( n \) for which \( K \) can be illuminated by \( n \) points (resp., directions). Furthermore, Boltyanski [40, 50] showed that \( I(K) = n \) if and only if the smallest number of smaller positive homothets of \( K \) that can cover \( K \) is \( n \). Thus, the Hadwiger Covering Conjecture can be reformulated as the Illumination Conjecture, which states that for any \( d \)-dimensional convex body \( K \) we have \( I(K) \leq 2^d \), and \( I(K) = 2^d \) only if \( K \) is an affine \( d \)-cube.

Despite the interest in these problems they have only been solved in general in two dimensions or for select few classes of convex bodies. We survey these advances in Chapter 2. This apparent difficulty has recently led to the introduction of quantitative versions of illumination and covering problems. For instance, it can be seen that in the definition of the illumination number \( I(K) \), the light sources can be taken arbitrarily far from \( K \). However, it seems natural to start with a relatively small number of light sources and quantify how far they need to be from \( K \) in order to illuminate it. This is the idea behind the illumination parameter \( \text{ill}(K) \) of an \( o \)-symmetric convex body \( K \) defined by Bezdek [18] as follows.

\[
\text{ill}(K) = \inf \left\{ \sum_i \|p_i\|_K : \{p_i\} \text{ illuminates } K, p_i \in \mathbb{R}^d \right\}.
\]

Clearly, \( I(K) \leq \text{ill}(K) \), for \( o \)-symmetric convex bodies. Several authors have investigated the illumination parameter of \( o \)-symmetric convex bodies [18, 21, 110, 127], determining exact values in several cases.

Inspired by the above quantification ideas, Swanepoel [159] introduced the covering parameter of a \( d \)-dimensional convex body to quantify its covering properties. This is given
by
\[ C(K) = \inf \left\{ \sum_i (1 - \lambda_i)^{-1} : K \subseteq \bigcup_i (\lambda_i K + t_i), 0 < \lambda_i < 1, t_i \in \mathbb{E}^d \right\} \]

Thus, large homothets are penalized in the same way as far away light sources are penalized in the definition of the illumination parameter. Swanepoel [159] obtained several results on \( C(\cdot) \) including an upper bound depending only on the dimension \( d \) of \( K \). However, not much is known about the covering parameter. For instance, we do not know whether \( C(\cdot) \) is lower or upper semicontinuous on \( K^d \) and the only known exact value is \( C(C^d) = 2^{d+1} \).

Moving on from covering to packing, the kissing number problem asks for the maximum number \( k(d) \) of non-overlapping translates of \( B^d \) that can touch \( B^d \) in \( \mathbb{E}^d \). Clearly, \( k(2) = 6 \). However, the value of \( k(3) \) remained a mystery for several hundred years and caused a famous argument between Newton and Gregory in the 17th century. To date, the only known kissing number values correspond to \( d = 2, 3, 4, 8, 24 \). For a survey of kissing numbers we refer the interested reader to [54].

Generalizing the kissing number, the Hadwiger number or the translative kissing number \( H(K) \) of a \( d \)-dimensional convex body \( K \) is the maximum number of non-overlapping translates of \( K \) that all touch \( K \) in \( \mathbb{E}^d \). Given the difficulty of the kissing number problem, determining Hadwiger numbers is highly nontrivial with few exact values known for \( d \geq 3 \). The best general upper and lower bounds on \( H(K) \) are due to Hadwiger [84] and Talata [160] respectively, and can be expressed as

\[ 2^{cd} \leq H(K) \leq 3^d - 1, \quad (1.1) \]

where \( c \) is an absolute constant, and equality holds in the right inequality if and only if \( K \) is an affine \( d \)-dimensional cube.

Let \( P \) be a packing of translates of a convex body \( K \) in \( \mathbb{E}^d \) (i.e., a family of non-overlapping translates of \( K \) in \( \mathbb{E}^d \)). The contact graph of \( P \) is the (simple) graph whose vertices correspond to the packing elements, with two vertices joined by an edge if and only if the two
corresponding packing elements touch each other. The number of edges of a contact graph is called the contact number of the underlying packing. The contact number problem asks for the largest number $c(K, n, d)$ of edges in any contact graph of a packing of $n$ translates of $K$ [29]. If $K = B^d$, we simply write $c(n, d)$ instead of $c(B^d, n, d)$. The problem of determining $c(K, n, d)$ is equivalent to the Erdős-type repeated shortest distance problem in normed spaces proposed by Ulam [55, 69], which asks for the largest number of repeated shortest distances among $n$ points in $(\mathbb{R}^d, \|\cdot\|_K)$. Another way to look at the contact number problem is to think of it as the discrete analogue of the densest packing problem. Moreover, it has been observed by materials scientists that at low temperatures, particles of self-assembling materials such as colloids tend to form clusters so as to maximize the contact number [5, 96]. The reader can refer to the very recent survey [29] on contact numbers and their applications for details.

The above connection with materials science also brings in crystallization, that is, the tendency of interacting particles to arrange themselves in periodic order [162]. Mathematically, a periodic order usually gives rise to some underlying lattice. For example, it has been proven [93] that at low temperatures, the ground-states of a two-dimensional sticky potential not only achieve maximum contact numbers, but also lie on a triangular lattice. The same is true for more general classes of potentials considered in [146] and [162]. What happens when we vary physical conditions is a wide-open question. One such condition is total separability, whose effect on crystallization will be discussed in Section 5.3.

A packing of translates of a convex domain $K$ in $\mathbb{E}^2$ is said to be totally separable if any two packing elements can be separated by a line of $\mathbb{E}^2$ disjoint from the interior of every packing element. This notion was introduced by G. Fejes Tóth and L. Fejes Tóth [75] and has attracted significant attention. We can define a totally separable packing of translates of a $d$-dimensional convex body $K$ in a similar way, by requiring any two packing elements to be separated by a hyperplane in $\mathbb{E}^d$ disjoint from the interior of every packing element. One can think of a totally separable packing as a packing with barriers. Practical examples
include packaging of identical fragile convex objects using cardboard separators and layered arrangements of interacting particles.

Contact graphs arising from translative packings of a convex body are examples of simple graphs. We now consider some classes of directed graphs and hypergraphs that will be studied in this thesis. In particular, we are interested in efficiently generating special types of directed graphs and hypergraphs, called tournaments and hypertournaments, with prescribed imbalances and scores, respectively.

A simple digraph is a digraph with no parallel arcs directed from a vertex $u$ to a vertex $v$ or loops directed from a vertex to itself. We call a simple digraph an oriented graph if it is without any pair of oppositely directed arcs. A tournament is a complete oriented graph in the sense that any pair of vertices in it is joined by an arc. The order of a tournament is the number of vertices in it. In a tournament, the score $s_i$ of a vertex $v_i$ is the number of arcs directed away from that vertex, that is, the out-degree of $v_i$. The score sequence of a tournament is formed by listing the scores in nondecreasing order. Let us write $[x_i]^n_1$ to denote a sequence with $n$ terms. Landau [115] gave a simple characterization of score sequences of tournaments.

**Theorem 1.1.1.** A sequence $[s_i]^n_1$ of non-negative integers in nondecreasing order is the score sequence of a tournament if and only if for every $I \subseteq \{1, 2, \ldots, n\}$,

$$\sum_{i \in I} s_i \geq \binom{|I|}{2}, \quad (1.2)$$

with equality when $|I| = n$, where $|I|$ is the cardinality of the set $I$.

Several proofs of Landau’s theorem have appeared over the years [8, 60, 80, 115, 163], and it continues to play a central role in the theory of tournaments and their generalizations. Brualdi and Shen [61] strengthened Landau’s theorem by deriving a set of inequalities that are individually stronger than inequalities (1.2), but are collectively equivalent to these inequalities.
The set of scores of vertices in a tournament is called the score set of the tournament. Reid [148] conjectured that any finite nonempty set $S$ of non-negative integers is the score set of some tournament. He gave a constructive proof of the conjecture for the cases $|S| = 1, 2, 3$, while Hager [86] settled the cases $|S| = 4, 5$. In 1986, Yao announced a nonconstructive proof of Reid’s conjecture by arithmetic arguments [169]. Pirzada and Naikoo [142] obtained a construction of a tournament with a given score set in the special case when the score increments are increasing. However, so far, no constructive proof has been found for Reid’s conjecture in general.

In a digraph, the imbalance of a vertex $v_i$ is defined as $t_i = d_i^+ - d_i^-$, where $d_i^+$ and $d_i^-$ are respectively the out-degree and in-degree of $v_i$. The imbalance sequence of a digraph is formed by listing the vertex imbalances in nonincreasing order. If $T$ is a tournament with imbalance sequence $[t_i]_1^n$, we say that $T$ realizes $[t_i]_1^n$. Mubayi, Will and West [138] gave Erdős–Gallai type necessary and sufficient conditions for a sequence of integers to be the imbalance sequence of a simple digraph.

**Theorem 1.1.2.** A sequence of integers $[t_i]_1^n$ with $t_1 \geq \cdots \geq t_n$ is an imbalance sequence of a simple digraph if and only if $\sum_{i=1}^j t_i \leq j(n-j)$, for $1 \leq j \leq n$ with equality when $j = n$.

On rearranging the imbalances in nondecreasing order, we obtain the equivalent inequalities $\sum_{i=1}^j t_i \geq j(n-j)$, for $1 \leq j \leq n$ with equality when $j = n$.

Koh and Ree [113] showed that if an additional parity condition is satisfied the sequence $[t_i]_1^n$ can be realized by a tournament. In fact, they proved the result in the more general setting of hypertournaments. The following corollary of Theorem 6 in [113] provides a characterization of imbalance sequences of tournaments.

**Theorem 1.1.3.** A nonincreasing sequence $[t_i]_1^n$ of integers is the imbalance sequence of a tournament if and only if $n - 1, t_1, \ldots, t_n$ have the same parity and

$$\sum_{i=1}^j t_i \leq j(n-j), \quad (1.3)$$
for } j = 1, \ldots, n \text{ with equality when } j = n.

In a digraph, the set of imbalances of the vertices is called its imbalance set \([143]\). In \([143]\) it is proved that if } P \text{ is a finite nonempty set of positive integers and } Q \text{ a finite nonempty set of negative integers, then there exists an oriented graph with imbalance set } P \cup Q. \text{ Due to the interest in Reid’s score set theorem, it is natural to ask if a similar result holds for imbalance sets of tournaments. Furthermore, since a constructive proof of Reid’s theorem has not yet been found, it would be interesting to look for an algorithm that generates a tournament from its imbalance set. Therefore, we pose the following decision and search problem.

**Definition 1.1.4 (Tournament Imbalance Set (TIS)).** Given a finite non-empty set } Z \text{ of integers, decide if } Z \text{ is the imbalance set of a tournament. If so, construct a tournament realizing } Z.

A } k\text{-hypergraph } H = (V, E) \text{ consists of a set } V \text{ of vertices and a set } E \text{ of edges that are subsets of } V, \text{ each subset containing exactly } k \text{ vertices. An oriented } k\text{-hypergraph is a } k\text{-hypergraph with all edges endowed with an orientation so that each edge is essentially an ordered } k\text{-tuple of vertices. The edges of an oriented hypergraph are generally called arcs. A } k\text{-hypertournament is a complete oriented } k\text{-hypergraph in the sense that for every set } X \text{ of } k \text{ vertices, it contains as an arc exactly one of the possible } k! \text{ orientations of } X.

Hypertournaments are generalizations of tournaments. One of the important problems in the theory of tournaments is to generate all labelled tournaments with a given score sequence. McKay and Wang [133] and Gao et al. [77] considered the problem of asymptotic enumeration of tournaments with a fixed score sequence. Kannan et al. [104] and McShine [134] addressed the problem of random generation of labelled tournaments with a given score sequence. However, so far, the corresponding problems for hypertournaments are still open, and our work is a first step in this direction.

In a } k\text{-hypertournament, the score } s_i \text{ (losing score } r_i \text{) of a vertex } v_i \text{ is the number of arcs containing } v_i \text{ in which } v_i \text{ is not the last element (in which } v_i \text{ is the last element). The total}
score \( t_i \) of \( v_i \) is defined as \( t_i = s_i - r_i \). The score sequence (resp. losing score sequence) is formed by listing the scores (resp. losing scores) in (usually) nondecreasing order, while the total score sequence is formed by listing the total scores in (usually) nonincreasing order. The following characterizations of scores and losing scores in \( k \)-hypertournaments are due to Zhou et al. \[170\] and generalize Landau’s theorem.

**Theorem 1.1.5.** Given non-negative integers \( n \) and \( k \) with \( n \geq k > 1 \), a nondecreasing sequence \( S = [s_i]_1^n \) of non-negative integers is the score sequence of some \( k \)-hypertournament if and only if for each \( I \subseteq \{1, 2, \ldots, n\} \) we have

\[
\sum_{i \in I} s_i \geq |I| \left( \frac{n-1}{k-1} \right) + \binom{n-|I|}{k} - \binom{n}{k},
\]

with equality when \( I = \{1, 2, \ldots, n\} \).

**Theorem 1.1.6.** Given two non-negative integers \( n \) and \( k \) with \( n \geq k > 1 \), a nondecreasing sequence \( R = [r_i]_1^n \) of non-negative integers is the losing score sequence of some \( k \)-hypertournament if and only if for each \( I \subseteq \{1, 2, \ldots, n\} \) we have

\[
\sum_{i \in I} r_i \geq \binom{|I|}{k},
\]

with equality when \( I = \{1, 2, \ldots, n\} \).

Koh and Ree \[113\] gave another proof of Theorem 1.1.6 based on finding a system of distinct representatives of a family of sets. A short proof of Theorem 1.1.6 can be seen in \[144\]. Some more results on the score and losing score sequences of \( k \)-hypertournaments are given in \[106\].

Zero-one matrices often arise as adjacency or incidence matrices of combinatorial structures. In particular, zero-one matrices with fixed row and column sums have several applications in the theory of graphs and hypergraphs \[58, 135\]. For details of combinatorial matrix theory, the interested reader is referred to the books by Bapat \[9\] and Brualdi \[59\].
The $(0,1)$-adjacency matrix of a tournament is called a tournament matrix. Kirkland and Shader [109] studied the spectral properties of tournament matrices. Some generalizations of tournament matrices can be seen in [108, 132, 137]. Koh and Ree [112, 113] defined a $k$-hypertournament matrix as the incidence matrix of a $k$-hypertournament. If $H = (V,E)$ is a $k$-hypertournament with $n$ vertices, the $k$-hypertournament matrix $M(H)$ corresponding to $H$ is defined as the $n \times \binom{n}{k}$ matrix whose $(i,j)$ entry is given by

$$m_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ appears in arc } e_j \text{ but not as the last vertex,} \\ -1 & \text{if } v_i \text{ appears in } e_j \text{ as the last vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

Several properties of $k$-hypertournament matrices are given in [113]. We summarize some of these in the following lemma which we shall use in the sequel.

**Lemma 1.1.7.** The column sum for any column of a $k$-hypertournament matrix $M(H)$ equals $k - 2$ while the sum along the $i^{th}$ row is equal to the total score $t_i$ of vertex $v_i$ of the $k$-hypertournament $H$. Furthermore, each row (respectively, column) contains exactly $\binom{n-1}{k-1}$ (respectively, $k - 1$) non-zero entries.

We now define the score, losing score and total score sequence of a $k$-hypertournament matrix.

**Definition 1.1.8.** Let $S = [s_i]^n_1$ be the score sequence of a $k$-hypertournament $H$ arranged arbitrarily and not necessarily in nondecreasing order. Then $[t_i]^n_1 = [s_i - r_i]^n_1 = [2s_i - \binom{n-1}{k-1}]^n_1$ is the total score sequence of $H$ not necessarily arranged in nonincreasing order. We denote by $\mathcal{M}_S$ the set of $k$-hypertournament matrices of order $n \times \binom{n}{k}$ whose $i^{th}$ row sums to $t_i$. For any $M \in \mathcal{M}_S$, we say that $S = [s_i]^n_1$ is the score sequence, $[r_i]^n_1$ is the losing score sequence and $[t_i]^n_1$ is the total score sequence of $M$. Thus $\mathcal{M}_S$ is the set of all $k$-hypertournament matrices with score sequence $S = [s_i]^n_1$. 

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Note that the concept of a \( k \)-hypertournament matrix differs from that of an \( h \)-hypertournament matrix as given in [108, 132]. An \emph{\( h \)-hypertournament matrix} is any square matrix \( A \) that satisfies \( A + A^T = hh^T - I \), where \( I \) is the identity matrix and \( h \) is a nonzero column matrix.

### 1.2 Outline of thesis and main results

The rest of this thesis is organized as follows. In Chapter 2 we provide a detailed account of the Covering Conjecture and its quantitative relatives. We discuss history, recent progress and possible approaches to attack the problems of homothetic covering and illumination of convex bodies. This is followed by Chapter 3 that builds on Chapter 2 and presents the results of our research on the quantitative homothetic covering of convex bodies. We come up with a more refined quantification of covering in terms of the covering index, denoted by \( \text{coin}(\cdot) \), with the Covering Conjecture as the eventual goal. We show that the covering index possesses several useful properties such as upper bounding several quantities associated with the covering and illumination of convex bodies, lower semicontinuity, compatibility with direct vector sum and Minkowski sum, a complete characterization of minimizers and the development of tools to compute its exact values for several convex bodies. Some of these are summarized below.

**Theorem 1.2.1.** Let \( d \) be any positive integer and

\[
\gamma_m(K) = \inf \left\{ \lambda > 0 : K \subseteq \bigcup_{i=1}^{m} (\lambda K + t_i), t_i \in \mathbb{E}^d, i = 1, \ldots, m \right\}.
\]

(i) Define \( I_K = \{ i : \gamma_i(K) \leq 1/2 \} \), for any \( d \)-dimensional convex body \( K \). If \( I_L \subseteq I_K \), for some \( K, L \in \mathcal{K}^d \), then \( \text{coin}(K) \leq \frac{2d_B(K, L)}{d_B(K, L)} \text{coin}(L) \leq d_B(K, L) \text{coin}(L) \).

(ii) The functional \( \text{coin} : \mathcal{K}^d \rightarrow \mathbb{R} \) is lower semicontinuous for all \( d \).

(iii) Define \( \mathcal{K}^{d*} := \{ K \in \mathcal{K}^d : \gamma_m(K) \neq 1/2, m \in \mathbb{N} \} \). Then the functional \( \text{coin} : \mathcal{K}^{d*} \rightarrow \mathbb{R} \) is
continuous for all \( d \).

**Theorem 1.2.2.**

(i) Let \( \mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n \) be a decomposition of \( \mathbb{E}^d \) into the direct vector sum of its linear subspaces \( \mathbb{L}_i \). Also let \( K_i \subseteq \mathbb{L}_i \) be convex bodies such that \( N_\lambda(K_i) \) denotes the minimum number of translates of \( \lambda K_i \) needed to cover \( K_i \), and \( \Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\} \). Then

\[
\max_{1 \leq i \leq n} \{\text{coin}(K_i)\} \leq \text{coin}(K_1 \oplus \cdots \oplus K_n) \leq \inf_{\lambda \leq 1} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda} \leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1 - \Gamma} < \prod_{i=1}^n \text{coin}(K_i).
\]

(1.4)

(ii) The first two upper bounds in (1.4) are tight. Moreover, the second inequality in (1.4) becomes an equality if any \( n - 1 \) of the \( K_i \)'s are tightly covered (see Definition 3.4.1).

(iii) Recall that \( \ell \in \mathbb{K}^1 \) denotes a line segment. If \( K \) is any convex body, then \( \text{coin}(K \oplus \ell) = 4 N_{1/2}(K) \).

(iv) Let the convex body \( K \) be the Minkowski sum of the convex bodies \( K_1, \ldots, K_n \in \mathbb{K}^d \) and \( \Gamma \) be as in part (i). Then

\[
\text{coin}(K) \leq \inf_{\lambda \leq 1} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda} \leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1 - \Gamma} < \prod_{i=1}^n \text{coin}(K_i).
\]

(1.5)

**Theorem 1.2.3.**

(i) Let \( d \) be any positive integer and \( K \in \mathbb{K}^d \). Then \( \text{coin}(C^d) = 2^{d+1} \leq \text{coin}(K) \) and thus, \( d \)-cubes minimize the covering index in all dimensions.

(ii) If \( K \) is a convex domain then \( \text{coin}(K) \leq \text{coin}(B^2) = 14 \).

Furthermore, the covering index gives rise to several open problems about the homothetic covering behavior of convex bodies in general, and \( d \)-dimensional balls and ball-polyhedra in particular. In Section 3.6, we discuss a variant of the covering index that is perhaps more natural, but possesses weaker properties. Finally, in Section 3.7, we obtain upper bounds on the covering and weak covering indices.
Chapter 4 provides a detailed treatment of the history, motivation and progress on the contact number problem in different settings. The focus is on recent advances as well as on applied aspects of the problem. In Chapter 5, we introduce the totally separable analogues of the Hadwiger number and the contact number for \(d\)-dimensional convex bodies, which we denote by \(H_{\text{sep}}(\cdot)\) and \(c_{\text{sep}}(\cdot, n, d)\), respectively and then mostly study them in the plane. Note that the quantity \(c_{\text{sep}}(n, d) = c_{\text{sep}}(B^d, n, d)\) was investigated in [36]. Some results obtained in that paper will be stated and used in the sequel. However, not much is known about the quantities \(H_{\text{sep}}(K)\) and \(c_{\text{sep}}(K, n, 2)\), even when \(K\) is a convex domain.

The main results of Chapter 5 can be summarized as below.

**Theorem 1.2.4.**

(A) \(H_{\text{sep}}(K) = 4\), for any smooth convex domain \(K\) in \(\mathbb{E}^2\).

(B) \(c_{\text{sep}}(K, n, 2) = \lceil 2n - 2\sqrt{n} \rceil\), for any smooth strictly convex domain \(K\) in \(\mathbb{E}^2\).

(C) Any packing of \(H_{\text{sep}}(K_0)\) translates of a smooth \(o\)-symmetric convex domain, all touching \(K_o\), together with \(K_o\) is a finite lattice packing (lying on an Auerbach lattice of \(K_o\)).

(D) Let \(R\) be a smooth Radon domain and let \(n = \ell(\ell+\epsilon)+k \geq 4\) be the unique decomposition of a positive integer \(n\) such that \(k, \ell\) and \(\epsilon\) are integers satisfying \(\epsilon \in \{0, 1\}\) and \(0 \leq k < \ell + \epsilon\). Suppose that \(P\) is a totally separable packing of \(n\) translates of \(R\) with \(c_{\text{sep}}(R, n, 2) = \lceil 2n - 2\sqrt{n} \rceil\) contacts. If \(k \neq 1\), then \(P\) is a finite lattice packing lying on an Auerbach lattice of \(R\), while if \(k = 1\), then all but at most one translate in \(P\) form a lattice packing on an Auerbach lattice of \(R\).

We prove parts (A) and (C) in Section 5.1. The tools used to achieve this include a characterization of total separability in smooth finite dimensional real normed spaces in terms of hemispherical caps on their unit balls, Birkhoff orthogonality and Auerbach bases in normed spaces. Part (B) is proved in Section 5.2 after a series of results giving separable contact numbers of smooth \(A\)- and \(B\)-domains followed by an approximation of smooth strictly convex domains by \(A\)-domains that preserves maximal separable contact number.
Theorem 1.2.5. Affine images of smooth strictly convex $A$-domains are dense (in the Hausdorff sense) in the space of smooth $o$-symmetric strictly convex domains. Moreover, given any smooth $o$-symmetric strictly convex domain $K_o$, we can obtain an affine image $A'$ of a smooth strictly convex $A$-domain $A$ such that the length of $\text{bd } A' \cap \text{bd } K_o$ can be made arbitrarily close to the length of $\text{bd } K_o$.

This part of the thesis makes heavy use of polyominoes in special type of lattices, which we call Auerbach lattices, and angle measures in normed planes. The proof of (D) is the subject of Section 5.3 which relies on some interesting properties of smooth Radon domains. The tools used in proving each statement are also discussed and developed in the corresponding sections. We note that Theorem 1.2.4 (A) and (B) respectively generalize the classical result $H_{sep}(B^2) = 4$ and the result $c_{sep}(n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor$ proved by Bezdek, Szalkai and Szalkai [36], respectively.

In Chapter 6, we solve the Tournament Imbalance Set (TIS) problem. First, in Section 6.1 we observe that the imbalance set of any tournament either consists of all even or all odd integers, and that the obvious necessary conditions for the existence of tournament imbalance sets are not sufficient. We then completely characterize the sets of odd integers that are imbalance sets of tournaments. In Section 6.2, we treat the case of even integers, which is more involved. We show that any set of even integers that contains at least one positive and at least one negative integer, or only consists of a single element 0, is the imbalance set of a partial tournament in which each vertex is joined to every other vertex except one. However, not all such sets are imbalance sets of tournaments. This is followed by necessary and sufficient conditions for a set of even integers to be a tournament imbalance set. We can summarize some of these results as follows.

Theorem 1.2.6.

(A) A finite nonempty set of odd integers is the imbalance set of a tournament if and only if it contains at least one positive and at least one negative integer.
(B) Let $Z = X \cup Y$ be a finite nonempty set of even integers, where $X$ is the set of non-negative integers and $Y$ is the set of negative integers in $Z$. Then $Z$ is the imbalance set of a tournament if and only if either $Z = \{0\}$ or both $X$ and $Y$ are nonempty and satisfy one of the following conditions:

(i) $0 \in X \cup Y$.

(ii) There exist an odd number of (not necessarily distinct) $x_{p_1}, \ldots, x_{p_{2r+1}} \in X$ and an even number of (not necessarily distinct) $-y_{q_1}, \ldots, -y_{q_{2s}} \in Y$ such that $\sum_{j=1}^{2r+1} x_{p_j} = \sum_{j=1}^{2s} y_{q_j}$.

(iii) There exist an odd number of (not necessarily distinct) $-y_{p_1}, \ldots, -y_{p_{2r+1}} \in Y$ and an even number of (not necessarily distinct) $x_{q_1}, \ldots, x_{q_{2s}} \in X$ such that $\sum_{j=1}^{2r+1} y_{p_j} = \sum_{j=1}^{2s} x_{q_j}$.

In Section 6.3, we describe the EQUAL-SUM SUBSETS (ESS) search problem that has been of great interest to computer scientists. We then define a new variant of this problem which we call EQUAL-SUM SEQUENCES (ESSeq) and show that ESSeq is NP-hard. Finally, in Section 6.4 we show how an ESSeq algorithm together with several other procedures can be used to solve TIS. Algorithm 6.4.2 determines if a set of integers is a tournament imbalance set and, in addition, generates a tournament realizing any such set.

Theorem 1.2.7. Let $Z = X \cup Y$ be a finite set of integers, with $X$ and $Y$ being disjoint sets of non-negative and negative integers in $Z$, respectively. Let $|X| = l$, $|Y| = m$, $L = \sum_{x \in X} x$, $M = \sum_{y \in Y} y$ and $n = lM + mL$. Then Algorithm 6.4.2 determines if $Z$ is the imbalance set of a tournament and, if so, constructs such a tournament in $O(2^{n \max\{l,m\}})$ time.

Note that here $l$ and $m$ determine the size of the input but $n$ is not necessarily polynomial in the size of the input. Instead, $n$ is polynomial in the numeric values of the input. We conclude Chapter 6 with a discussion of extremal cases and determine upper bounds on the minimal order of a tournament realizing an imbalance set.
In Chapter 7 we present some structural results on hypetournaments and their regularity (Section 7.1), and then consider the problem of uniformly sampling labelled $k$-hypetournaments with a fixed score (losing score) sequence. Combinatorial literature abounds with problems of randomly generating all instances of a combinatorial structure sharing some property. For example, the problem of generating all simple graphs with a given degree sequence is known as the degree sequence problem and is of great interest to mathematicians as well as researchers in other disciplines [67]. Since the number of such instances can be exponentially large, randomized algorithms are needed that can sample these instances uniformly and efficiently. Herein lies a problem, as often one has to compromise on the efficiency or the uniformity of the procedure, just because it seems impossible to achieve both [10, 67]. We show that both can be achieved for hypetournaments.

We begin by sampling through $k$-hypetournament matrices with the same score sequence. In Section 7.2, we define switchable configurations and the operation of switching that converts a $k$-hypetournament matrix into another $k$-hypetournament matrix. We show that every $k$-hypetournament matrix has a switchable configuration and that switchings preserve the score sequence. An algorithm is presented that lists all switchable configurations in a $k$-hypetournament matrix. Section 7.3 gives an algorithm to generate a $k$-hypetournament matrix corresponding to a given score sequence. The algorithm is based on the proof of Theorem 1.1.6 given in [113] that guarantees the existence of a system of distinct representatives of a special family of sets. We then use the switching operation to construct a Markov chain $M_{mc}$ on the set $\mathcal{M}_S$ of all $k$-hypetournament matrices with a given score sequence $S$. We prove that $M_{mc}$ is ergodic with a uniform stationary distribution. Some preliminaries on Markov chains are included in Section 7.3. In Section 7.4, we define the mixing time of a Markov chain based on [156]. We show that the mixing time of $M_{mc}$ is bounded above by a polynomial in the size of the input (score sequence) and hence $M_{mc}$ is ‘rapidly mixing’.

**Theorem 1.2.8.** The mixing time of $M_{mc}$ satisfies $\tau_s(\epsilon) \leq 128m^3n^2(\ln k + \ln \frac{1}{\epsilon})$, where $n$ is the number of vertices and $m$ is the number of arcs of a $k$-hypetournament having the
In Section 7.5, we discuss the construction of a $k$-hypertournament associated with a $k$-hypertournament matrix and define a Markov chain $\mathcal{H}_{mc}$ on the set $\mathcal{H}_S$ of all labelled $k$-hypertournaments with score sequence $S$. We show that the ergodicity, the uniform stationary distribution and rapid mixing of $\mathcal{H}_{mc}$ follow from the corresponding properties of $\mathcal{M}_{mc}$.

**Theorem 1.2.9.** The Markov chain $\mathcal{H}_{mc}$ is rapid mixing and ergodic with uniform stationary distribution.

Finally, Chapter 8 recaps the main contributions of this thesis and presents open problems for future work.
Chapter 2

Homothetic covering and illumination of convex bodies

At a first glance, the problem of illuminating the boundary of a convex body by external light sources and the problem of covering a convex body by its smaller positive homothetic copies appear to be quite different. They are in fact two sides of the same coin and give rise to one of the important longstanding open problems in discrete geometry, namely, the Illumination Conjecture. In this chapter, we survey the activity in the areas of discrete geometry, computational geometry and geometric analysis motivated by this conjecture. This lays the groundwork for Chapter 3 which describes a new approach to the problem.

2.1 Shedding some ‘light’

“… $N_k$ bezeichne die kleinste natürliche Zahl, für welche die nachfolgende Aussage richtig ist: Ist $A$ ein eigentlicher konvexer Körper des $k$-dimensionalen euklidischen Raumes, so gibt es $n$ mit $A$ translations-gleiche Körper $A_i$ mit $n \leq N_k$ derart, dass jeder Punkt von $A$ ein innerer Punkt der Vereinigungsmenge $\bigcup_i A_i$ ist, … Welchen Wert hat $N_k$ für $k \geq 3$?” [83]

The above statement roughly translates to “Let $N_k$ denote the smallest natural number such
that any \( k \)-dimensional convex body can be covered by the interior of a union of at the most \( N_k \) of its translates. What is \( N_k \) for \( k \geq 3 \)?” When Hadwiger raised this question in 1957 he probably did not imagine that it would remain unresolved half a century later and become a central problem in discrete geometry. However, he was not the first one to study this particular problem. In fact, its earliest occurrence can be traced back to Levi’s 1955 paper [122], who formulated and settled the 2-dimensional case of the problem. Later in 1960, the question was restated by Gohberg and Markus [79] – who worked on the problem independently without knowing about the work of Levi and Hadwiger [46] – in terms of covering by homothetic copies. The equivalence of both formulations is relatively easy to check and details appear in Section 34 of [50].

**Conjecture 2.1.1 (Covering Conjecture).** We can cover any \( d \)-dimensional convex body by \( 2^d \) or fewer of its smaller positive homothetic copies in Euclidean \( d \)-space, \( d \geq 3 \). Furthermore, \( 2^d \) homothetic copies are required only if the body is an affine \( d \)-cube.

In other words, the Covering Conjecture states that for any convex body \( \mathbf{K} \subseteq \mathbb{E}^d \), there exist \( \mathbf{t}_i \in \mathbb{E}^d \) and \( 0 < \lambda_i < 1 \), for \( i = 1, \ldots, 2^d \), such that

\[
\mathbf{K} \subseteq \bigcup_{i=1}^{2^d} (\mathbf{t}_i + \lambda_i \mathbf{K}).
\]
The same conjecture has also been referred to in the literature as the Levi–Hadwiger Conjecture, Gohberg–Markus Covering Conjecture and Hadwiger Covering Conjecture. The condition $d \geq 3$ has been added as the statement is known to be true in the plane [85, 122].

A light source at a point $p$ outside a convex body $K \subset \mathbb{R}^d$, illuminates a point $x$ on the boundary of $K$ if the halfline originating from $p$ and passing through $x$ intersects the interior of $K$ at a point not lying between $p$ and $x$. The set of points $\{p_i : i = 1, \ldots, n\}$ in the exterior of $K$ is said to illuminate $K$ if every boundary point of $K$ is illuminated by some $p_i$. The illumination number $I(K)$ of $K$ is the smallest $n$ for which $K$ can be illuminated by $n$ point light sources.

One can also consider illumination of $K \subset \mathbb{R}^d$ by parallel beams of light. Let $S^{d-1}$ be the unit sphere centered at the origin $o$ of $\mathbb{R}^d$. We say that a point $x$ on the boundary of $K$ is illuminated in the direction $v \in S^{d-1}$ if the halfline originating from $x$ and with direction vector $v$ intersects the interior of $K$.

The former notion of illumination was introduced by Hadwiger [85], while the latter notion is due to Boltyanski [40]. It may not come as a surprise that the two concepts are equivalent in the sense that a convex body $K$ can be illuminated by $n$ point sources if and only if it
Figure 2.3: (a) Illuminating a boundary point $x$ of $K \subseteq \mathbb{E}^d$ by a direction $v \in S^{d-1}$, (b) $I(K) = 3$.

can be illuminated by $n$ directions. Though less obvious, any covering of $K$ by $n$ smaller positive homothetic copies corresponds to illuminating $K$ by $n$ points (or directions) and vice versa (see [50] for details). Therefore, the following Illumination Conjecture [40, 50, 85] of Hadwiger and Boltyanski is equivalent to the Covering Conjecture.

**Conjecture 2.1.2 (Illumination Conjecture).** The illumination number $I(K)$ of any $d$-dimensional convex body $K$, $d \geq 3$, is at most $2^d$ and $I(K) = 2^d$ only if $K$ is an affine $d$-cube.

The conjecture also asserts that affine images of $d$-cubes are the only extremal bodies. The conjectured bound of $2^d$ results from the $2^d$ vertices of an affine $d$-cube, each requiring a different light source to be illuminated. In the sequel, we use the titles Covering Conjecture, Hadwiger Covering Conjecture and Illumination Conjecture interchangeably, shifting between the covering and illumination paradigms as convenient.

We have so far seen three equivalent formulations of the Illumination Conjecture, but there are more. In fact, it is perhaps an indication of the richness of this problem that renders it to be studied from many different perspectives, each with its own intuitive significance. We state one more equivalent form found independently by P. Soltan and V. Soltan [158], who formulated it for the $\mathcal{o}$-symmetric case only, and Bezdek [16, 17].

**Conjecture 2.1.3 (Separation Conjecture).** Let $K$ be an arbitrary convex body in $\mathbb{E}^d$,
$d \geq 3$, and $o$ be an arbitrary interior point of $K$. Then there exist $2^d$ hyperplanes of $\mathbb{E}^d$ such that each intersection of $K$ with a supporting hyperplane, called a face of $K$, can be strictly separated from $o$ by at least one of the $2^d$ hyperplanes. Furthermore, $2^d$ hyperplanes are needed only if $K$ is the convex hull of $d$ linearly independent line segments which intersect at the common relative interior point $o$.

Over the years, the Illumination Conjecture has inspired a vast body of research in convex and discrete geometry, computational geometry and geometric analysis. There exist some nice surveys on the topic such as the papers [19, 127], the corresponding chapters of the books [21, 50] and the very recent survey by Bezdek and the author [30].

We organize the rest of this Chapter as follows. Section 2.2 gives a brief overview of the progress on the Illumination Conjecture. In Section 2.3, we explore the known important quantitative versions of the problem. Finally in Section 2.4, we present Zong’s computer-assisted approach [171] for possibly resolving the Illumination Conjecture in low dimensions.

2.2 Progress on the Illumination Conjecture

2.2.1 Results in $\mathbb{E}^3$ and $\mathbb{E}^4$

Despite its intuitive richness, the Illumination Conjecture has been notoriously difficult to crack even in the first nontrivial case of $d = 3$. The closest anything has come is the proof announced by Boltyanski [43] for the 3-dimensional case. Unfortunately, the proof turned out to have gaps that remain to date. Later, Boltyanski [44] modified his claim to the following.

**Theorem 2.2.1.** Let $K$ be a convex body of $\mathbb{E}^3$ with $md(K) = 2$. Then $I(K) \leq 6$.

Here $md$ is a functional introduced by Boltyanski in [41] and defined as follows for any $d$-dimensional convex body: Let $K \subseteq \mathbb{E}^d$ be a convex body. Then $md(K)$ is the greatest integer $m$ for which there exist $m+1$ regular boundary points of $K$ such that the outward unit normals $v_0, \ldots, v_m$ of $K$ at these points are minimally dependent, i.e., they are the vertices
of an $m$-dimensional simplex that contains the origin in its relative interior. In fact, it is
proved in [41] that $\text{md}(K) = \text{him}(K)$ (see the discussion preceding Theorem 2.2.16) holds
for any convex body $K$ of $\mathbb{E}^d$ and therefore one can regard Theorem 2.2.1 as an immediate
corollary of Theorem 2.2.16 in Section 2.2.2.

So far the best upper bound on the illumination number in three dimensions is due to
Papadoperakis [141].

**Theorem 2.2.2.** The illumination number of any convex body in $\mathbb{E}^3$ is at most 16.

However, there are partial results that establish the validity of the conjecture for some
large classes of convex bodies. Often these classes of convex bodies have some underlying
symmetry. Here we list some such results. Recall that a convex body $K$ is said to be centrally
symmetric if it has a point of symmetry. Lassak [116] proved that under the assumption of
central symmetry, the illumination conjecture holds in three dimensions.

**Theorem 2.2.3.** If $K$ is a centrally symmetric convex body in $\mathbb{E}^3$, then $I(K) \leq 8$.

A convex polyhedron $P$ is said to have affine symmetry if the affine symmetry group of $P$
consists of the identity and at least one other affinity of $\mathbb{E}^3$. Bezdek obtained the following
result [16].

**Theorem 2.2.4.** If $P$ is a convex polyhedron of $\mathbb{E}^3$ with affine symmetry, then the illumi-
nation number of $P$ is at most 8.

Dekster [66] extended Theorem 2.2.4 from polyhedra to convex bodies with plane sym-
metry. Note that a convex body $K$ is symmetric about a plane $p$ if a reflection across $p$
leaves $K$ unchanged.

**Theorem 2.2.5.** If $K$ is a convex body symmetric about a plane in $\mathbb{E}^3$, then $I(K) \leq 8$.

It turns out that for 3-dimensional bodies of constant width – that is bodies whose width,
measured by the distance between two opposite parallel hyperplanes touching its boundary,
is the same regardless of the direction of those two parallel planes – we get an even better bound.

**Theorem 2.2.6.** *The illumination number of any convex body of constant width in \( \mathbb{E}^3 \) is at most 6.*

Proofs of the above theorem have appeared in several papers [33, 120, 166]. It is, in fact, reasonable to conjecture the following even stronger result.

**Conjecture 2.2.7.** *The illumination number of any convex body of constant width in \( \mathbb{E}^3 \) is exactly 4.*

The above conjecture, if true, would provide a new proof of Borsuk’s conjecture [52] in dimension three, which states that any set of unit diameter in \( \mathbb{E}^3 \) can be partitioned into at most four subsets of diameter less than one. We remark that although it is false in general [103], Borsuk’s conjecture has a long and interesting history of its own and the reader can look up [45, 50, 82] for detailed discussions.

Now let us consider the state of the Illumination Conjecture in \( \mathbb{E}^4 \). It is well known that neighbourly \( d \)-polytopes have the maximum number of facets among \( d \)-polytopes with a fixed number of vertices (for more details on this see, for example, [39]). Thus, it is natural to investigate the Separation Conjecture for neighbourly \( d \)-polytopes (see also Theorem 2.2.18). Since interesting neighbourly \( d \)-polytopes exist only in \( \mathbb{E}^d \) for \( d \geq 4 \), it is particularly natural to first restrict our attention to neighbourly 4-polytopes. Starting from a cyclic 4-polytope, the sewing procedure of Shemer (for details see [39]) produces an infinite family of neighbourly 4-polytopes each of which is obtained from the previous one by adding one new vertex in a suitable way. Neighbourly 4-polytopes obtained from a cyclic 4-polytope by a sequence of sewings are called *totally-sewn*. The main result of the very recent paper [39] of Bisztriczky and Fodor is a proof of the Separation Conjecture for totally-sewn neighbourly 4-polytopes.

**Theorem 2.2.8.** *Let \( P \) be an arbitrary totally-sewn neighbourly 4-polytope in \( \mathbb{E}^4 \), and \( o \) be an arbitrary interior point of \( P \). Then there exist 16 hyperplanes of \( \mathbb{E}^4 \) such that each face*
of $P$, can be strictly separated from $o$ by at least one of the 16 hyperplanes.

However, Bisztriczky [38] conjectures the following stronger result.

**Conjecture 2.2.9.** Let $P$ be an arbitrary totally-sewn neighbourly 4-polytope in $\mathbb{E}^4$, and $o$ be an arbitrary interior point of $P$. Then there exist 9 hyperplanes of $\mathbb{E}^4$ such that each face of $P$, can be strictly separated from $o$ by at least one of the 9 hyperplanes.

### 2.2.2 General results

Before we state results on the illumination number of convex bodies in $\mathbb{E}^d$, we take a little detour. We need Rogers’ estimate [149] of the infimum $\theta(K)$ of the covering density of $\mathbb{E}^d$ by translates of the convex body $K$, namely, for $d \geq 2$, \(^1\)

$$\theta(K) \leq d(ln d + \ln \ln d + 5)$$

and the Rogers–Shephard inequality [151]

$$\text{vol}_d(K - K) \leq \binom{2d}{d} \text{vol}_d(K)$$

on the $d$-dimensional volume $\text{vol}_d(\cdot)$ of the difference body $K - K = K + (-K)$ of $K$.

It was rather a coincidence, at least from the point of view of the Illumination Conjecture, when in 1964 Erdős and Rogers [71] proved the following theorem. In order to state their theorem in a proper form we need to introduce the following notion. If we are given a covering of a space by a system of sets, the *star number* of the covering is the supremum, over sets of the system, of the cardinals of the numbers of sets of the system meeting a set of the system (see [71]). On the one hand, the standard Lebesgue brick-laying construction provides an example, for each positive integer $d$, of a lattice covering of $\mathbb{E}^d$ by closed cubes with star number $2^{d+1} - 1$. On the other hand, Theorem 1 of [71] states that the star number

\(^1\)The bound on $\theta(K)$ has been improved to $\theta(K) \leq d \ln d + d \ln \ln d + d + o(d)$ by G. Fejes Tóth [74].
of a lattice covering of $\mathbb{E}^d$ by translates of a centrally symmetric convex body is always at least $2^{d+1} - 1$. However, from our point of view, the main result of [71] is the one under Theorem 2 which (combined with some observations from [70] and the Rogers–Shephard inequality [149]) reads as follows.

**Theorem 2.2.10.** Let $K$ be a convex body in the $d$-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. Then there exists a covering of $\mathbb{E}^d$ by translates of $K$ with star number at most

$$\frac{\text{vol}_d(K - K)}{\text{vol}_d(K)}(d \ln d + d \ln \ln d + 5d + 1) \leq \left(\frac{2d}{d}\right)(d \ln d + d \ln \ln d + 5d + 1).$$

Moreover, for sufficiently large $d$, $5d$ can be replaced by $4d$.

The periodic and probabilistic construction on which Theorem 2.2.10 is based also gives the following.

**Corollary 2.2.11.** If $K$ is an arbitrary convex body in $\mathbb{E}^d$, $d \geq 2$, then

$$I(K) \leq \frac{\text{vol}_d(K - K)}{\text{vol}_d(K)}(\ln d + \ln \ln d + 5) \leq \left(\frac{2d}{d}\right)(\ln d + \ln \ln d + 5) = O(4^d \sqrt{d \ln d}).$$ (2.2)

Moreover, for sufficiently large $d$, $5d$ can be replaced by $4d$.

Note that the bound given in Corollary 2.2.11 can also be obtained from the more general result of Rogers and Zong [152], which states that for $d$-dimensional convex bodies $K$ and $L$, $d \geq 2$, one can cover $K$ by $N(K, L)$ translates $^2$ of $L$ such that

$$N(K, L) \leq \frac{\text{vol}_d(K - L)}{\text{vol}_d(L)}\theta(L).$$

For the sake of completeness we also mention the inequality

$$I(K) \leq (d + 1)d^{d-1} - (d - 1)(d - 2)^{d-1}$$

$^2N(K, L)$ is called the covering number of $K$ by $L$. 

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due to Lassak [119], which is valid for an arbitrary convex body $K$ in $\mathbb{E}^d$, $d \geq 2$, and is (somewhat) better than the estimate of Corollary 2.2.11 for some small values of $d$.

Since, for a centrally symmetric convex body $K$, $\frac{\text{vol}(K-K)}{\text{vol}(K)} = 2^d$, we have the following improved upper bound on the illumination number of such convex bodies.

**Corollary 2.2.12.** If $K$ is a centrally symmetric convex body in $\mathbb{E}^d$, $d \geq 2$, then

$$I(K) \leq \frac{\text{vol}_d(K-K)}{\text{vol}_d(K)}d(\ln d + \ln \ln d + 5) = 2^d d(\ln d + \ln \ln d + 5) = O(2^d d \ln d).$$  \hspace{1cm} (2.3)

The above upper bound is fairly close to the conjectured value of $2^d$. However, most convex bodies are far from being symmetric and so, in general, one may wonder whether the Illumination Conjecture is true at all, especially for large $d$. Thus, it was important progress when Schramm [154] managed to prove the Illumination Conjecture for all convex bodies of constant width in all dimensions at least 16. In fact, he proves the following inequality.

**Theorem 2.2.13.** If $W$ is an arbitrary convex body of constant width in $\mathbb{E}^d$, $d \geq 3$, then

$$I(W) \leq 5d \sqrt{d}(4 + \ln d) \left( \frac{3}{2} \right)^{\frac{d}{2}}.$$

By taking a closer look of Schramm’s elegant paper [154] and making the necessary modifications, Bezdek [22] somewhat improved the upper bound of Theorem 2.2.13, but more importantly he succeeded in extending that estimate to the following family of convex bodies (called the family of *fat spindle convex bodies*) that is much larger than the family of convex bodies of constant width. Thus, we have the following generalization of Theorem 2.2.13 proved in [22].

**Theorem 2.2.14.** Let $X \subset \mathbb{E}^d$, $d \geq 3$, be an arbitrary compact set with $\text{diam}(X) \leq 1$ and let $B[X]$ be the intersection of the closed $d$-dimensional unit balls centered at the points of $X$. Then

$$I(B[X]) < 4 \left( \frac{\pi}{3} \right)^{\frac{d}{2}} d^{\frac{3}{2}} (3 + \ln d) \left( \frac{3}{2} \right)^{\frac{d}{2}} < 5d^{\frac{3}{2}} (4 + \ln d) \left( \frac{3}{2} \right)^{\frac{d}{2}}.$$
On the one hand, \(4 \left(\frac{\pi}{3}\right)^{\frac{1}{2}} \cdot d^{\frac{3}{2}} (3 + \ln d) \left(\frac{3}{2}\right)^{\frac{d}{2}} < 2^d\) for all \(d \geq 15\). (Moreover, for every \(\epsilon > 0\), if \(d\) is sufficiently large, then \(I(\mathcal{B}[X]) < (\sqrt{1.5} + \epsilon)^d = (1.224\ldots + \epsilon)^d\).) On the other hand, based on the elegant construction of Kahn and Kalai [103], it is known (see [1]) that if \(d\) is sufficiently large, then there exists a finite subset \(X''\) of \(\{0, 1\}^d\) in \(\mathbb{E}^d\) such that any partition of \(X''\) into parts of smaller diameter requires more than \((1.2)^{\sqrt{d}}\) parts. Let \(X'\) be the (positive) homothetic copy of \(X''\) having unit diameter and let \(X\) be the (not necessarily unique) convex body of constant width one containing \(X'\). Then it follows via standard arguments that \(I(\mathcal{B}[X]) > (1.2)^{\sqrt{d}}\) with \(X = \mathcal{B}[X]\).

A convex polytope is called a belt polytope if to each side of any of its 2-faces there exists a parallel (opposite) side on the same 2-face. This class of polytopes is wider than the class of zonotopes. Moreover, it is easy to see that any convex body of \(\mathbb{E}^d\) can be represented as a limit of a convergent sequence of belt polytopes with respect to the Hausdorff metric. The following theorem on belt polytopes was proved by Martini in [126]. The result that it extends to the class of convex bodies, called belt bodies (including zonoids), is due to Boltyanski [47, 42, 50]. (See also [48] for a somewhat sharper result on the illumination numbers of belt bodies.)

**Theorem 2.2.15.** Let \(P\) be an arbitrary \(d\)-dimensional belt polytope (resp., belt body) different from a parallelotope in \(\mathbb{E}^d\), \(d \geq 2\). Then

\[
I(P) \leq 3 \cdot 2^{d-2}.
\]

Now, let \(K\) be an arbitrary convex body in \(\mathbb{E}^d\) and let \(\mathcal{T}(K)\) be the family of all translates of \(K\) in \(\mathbb{E}^d\). The Helly dimension \(\text{him}(K)\) of \(K\) ([157]) is the smallest integer \(h\) such that for any finite family \(\mathcal{F} \subseteq \mathcal{T}(K)\) with cardinality greater than \(h + 1\) the following assertion holds: if every \(h + 1\) members of \(\mathcal{F}\) have a point in common, then all the members of \(\mathcal{F}\) have a point in common. As is well known, \(1 \leq \text{him}(K) \leq d\). Using this notion, Boltyanski [44] gave a proof of the following theorem.
Theorem 2.2.16. Let $K$ be a convex body with $\text{him}(K) = 2$ in $\mathbb{E}^d$, $d \geq 3$. Then

$$I(K) \leq 2^d - 2^{d-2}.$$ 

In fact, in [44] Boltyanski conjectures the following more general inequality.

Conjecture 2.2.17. Let $K$ be a convex body with $\text{him}(K) = h > 2$ in $\mathbb{E}^d$, $d \geq 3$. Then

$$I(K) \leq 2^d - 2^{d-h}.$$ 

Bezdek and Bisztriczky gave a proof of the Illumination Conjecture for the class of dual cyclic polytopes in [24]. Their upper bound for the illumination numbers of dual cyclic polytopes has been improved by Talata in [161]. So, we have the following statement.

Theorem 2.2.18. The illumination number of any $d$-dimensional dual cyclic polytope is at most $\frac{(d+1)^2}{2}$, for all $d \geq 2$.

2.3 Quantifying illumination and covering

It can be seen that in the definition of illumination number $I(K)$, the distance of light sources from $K$ plays no role whatsoever. Starting with a relatively small number of light sources, it makes sense to quantify how far they need to be from $K$ in order to illuminate it. This is the idea behind the illumination parameter as defined by Bezdek [18].

Let $K$ be an $o$-symmetric convex body. Then the norm $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$ provides a good estimate of how far a point $x$ is from $K$. The illumination parameter $\text{ill}(K)$ of an $o$-symmetric convex body $K$ estimates how well $K$ can be illuminated by relatively few point sources lying as close to $K$ on average as possible.

$$\text{ill}(K) = \inf \left\{ \sum_i \|p_i\|_K : \{p_i\} \text{ illuminates } K, p_i \in \mathbb{E}^d \right\},$$
Clearly, \( I(K) \leq \text{ill}(K) \) holds for any \( o \)-symmetric convex body \( K \). In the papers [25, 110], the illumination parameters of \( o \)-symmetric Platonic solids have been determined. In [18], a tight upper bound was obtained for the illumination parameter \( o \)-symmetric convex domains.

**Theorem 2.3.1.** If \( K \) is an \( o \)-symmetric convex domain, then \( \text{ill}(K) \leq 6 \) with equality for any affine regular convex hexagon.

The corresponding problem in dimension 3 and higher is wide open. The following conjecture is due to Kiss and de Wet [110].

**Conjecture 2.3.2.** The illumination parameter of any \( o \)-symmetric 3-dimensional convex body is at most 12.

However, for smooth \( o \)-symmetric convex bodies in any dimension \( d \geq 2 \), Bezdek and Litvak [34] found an upper bound, which was later improved to the following asymptotically sharp bound by Gluskin and Litvak [78].

**Theorem 2.3.3.** For any smooth \( o \)-symmetric \( d \)-dimensional convex body \( K \),

\[
\text{ill}(K) \leq 24d^{3/2}.
\]

Translating the above quantification ideas from illumination into the setting of covering, Swanepoel [159] introduced the covering parameter of a convex body as follows.

\[
C(K) = \inf \left\{ \sum_i (1 - \lambda_i)^{-1} : K \subseteq \bigcup_i (\lambda_i K + t_i), 0 < \lambda_i < 1, t_i \in \mathbb{R}^d \right\}.
\]

Thus, large homothets are penalized in the same way as the far off light sources are penalized in the definition of illumination parameter. Note that here \( K \) need not be \( o \)-symmetric. In the same paper, Swanepoel obtained the following Rogers-type upper bounds on \( C(K) \) when \( d \geq 2 \). Here \( e \) denotes the base of natural logarithm.
Theorem 2.3.4.

\[ C(K) < \begin{cases} 
  e^{2d}(d+1)(\ln d + \ln \ln d + 5) = O(2^d d^2 \ln d), & \text{if } K \text{ is } o\text{-symmetric}, \\
  e^{\left(\frac{2d}{d}\right)}d(d+1)(\ln d + \ln \ln d + 5) = O(4^d d^{3/2} \ln d), & \text{otherwise.}
\end{cases} \tag{2.4} \]

He further showed that if \( K \) is \( o \)-symmetric, then

\[ \text{ill}(K) \leq 2C(K), \tag{2.5} \]

and therefore, \( \text{ill}(K) = O(2^d d^2 \ln d) \).

Based on the above results, it is natural to study the following quantitative analogue of the illumination conjecture that was proposed by Swanepoel [159].

**Conjecture 2.3.5 (Quantitative Illumination Conjecture).** For any \( o \)-symmetric \( d \)-dimensional convex body \( K \), \( \text{ill}(K) = O(2^d) \).

Before proceeding further, we introduce some terminology and notations. Let us use \( \mathcal{K}^d \) and \( \mathcal{C}^d \) respectively to denote the set of all \( d \)-dimensional convex bodies and the set of all such bodies that are \( o \)-symmetric. We consider some of the important properties of the illumination number and the covering parameter as functionals defined on \( \mathcal{K}^d \) and the illumination parameter as a functional on \( \mathcal{C}^d \). The first observation is that the three quantities are affine invariants (as are several other quantities dealing with the covering and illumination of convex bodies). That is, if \( A : \mathbb{E}^d \to \mathbb{E}^d \) is an affine transformation and \( K \) is any \( d \)-dimensional convex body, then \( I(K) = I(A(K)) \), \( \text{ill}(K) = \text{ill}(A(K)) \) and \( C(K) = C(A(K)) \). Due to this affine invariance, whenever we refer to a convex body \( K \), whatever we say about the covering and illumination of \( K \) is true for all affine images of \( K \). For instance, we can use the symbol \( B^d \) to denote a \( d \)-dimensional unit ball as well as its affine images called ellipsoids.

In this thesis, \( \mathcal{K}^d \) (resp., \( \mathcal{C}^d \)) is considered as a metric space under the Banach–Mazur
distance. Since continuity of a functional can provide valuable insight into its behaviour, it is of considerable interest to check the continuity of $I(\cdot)$, $\text{ill}(\cdot)$ and $C(\cdot)$. Unfortunately, by Example 2.3.6, the first two quantities are known to be discontinuous, while nothing is known about the continuity of the third.

**Example 2.3.6** (Smoothed cubes and spiky balls). Let $\epsilon > 0$. In $K^d$, consider a sequence $(\mathbf{C}_n = \mathbf{C}^d + \frac{1}{n}\epsilon)_{n \in \mathbb{N}}$ of ‘smoothed’ $d$-dimensional cubes that approaches $\mathbf{C}^d$ in the Banach–Mazur sense. Since the smoothed cubes are smooth convex bodies, all the terms of the sequence have illumination number $d + 1$. However, $I(\mathbf{C}^d) = 2^d$, which shows that $I(\cdot)$ is not continuous.

Recently, Naszódi [140] constructed a class of $d$-dimensional $o$-symmetric bodies, that he refers to as ‘spiky balls’. Pick $N$ points $\mathbf{x}_1, \ldots, \mathbf{x}_N$ independently and uniformly with respect to the Haar probability measure on the $(d - 1)$-dimensional unit sphere $S^{d-1}$ centred at the origin $o$. Then a spiky ball corresponding to a real number $D > 1$ is defined as

$$K = \text{conv} \left( \{\pm \mathbf{x}_i : i = 1, \ldots, N\} \cup \frac{1}{D} \mathbf{B}^d \right).$$

Straightaway we observe that $K$ is $o$-symmetric and satisfies $d_{BM}(K, \mathbf{B}^d) < D$. Naszódi showed that $I(K) \geq c^d$, where $c > 1$ is a constant depending on $d$ and $D$. Thus, we have a sequence of spiky balls approaching $\mathbf{B}^d$ in Banach–Mazur distance such that each spiky ball has an exponential illumination number. Since by Theorem 2.3.3, $\text{ill}(\mathbf{B}^d) = O(d^{3/2})$ and $\text{ill}(K) \geq I(K)$ we see that $\text{ill}(\cdot)$ is not continuous.

We can state the following about the continuity of $I(\cdot)$ [50].

**Theorem 2.3.7.** The functional $I(\cdot)$ is upper semicontinuous on $K^d$, for all $d \geq 2$.

Despite the usefulness of the covering parameter, not much is known about it. For instance, we do not know whether $C(\cdot)$ is lower or upper semicontinuous on $K^d$ and the only known exact value is $C(\mathbf{C}^d) = 2^{d+1}$. Thus, there is a need to propose a more refined
quantitative version of homothetic covering for convex bodies. Chapter 3 describes how we address this need.

2.4 A computer-based approach

Given a positive integer $m$, Lassak [117] introduced the $m$-covering number of a convex body $K$ as the minimal positive homothety ratio needed to cover $K$ by $m$ positive homothets. That is,

$$\gamma_m(K) = \inf \left\{ \lambda > 0 : K \subseteq \bigcup_{i=1}^{m} (\lambda K + t_i), t_i \in \mathbb{E}^d, i = 1, \ldots, m \right\}.$$

Lassak showed that the $m$-covering number is well-defined and studied the special case $m = 4$ for convex domains. It should be noted that special values of this quantity had been considered by several authors in the past. For instance, Bezdek showed that $\gamma_5(B^2) = 0.609382\ldots$ and $\gamma_6(B^2) = 0.555905\ldots$ [14, 15].

Zong [171] studied $\gamma_m : K^d \to \mathbb{R}$ as a functional and proved it to be uniformly continuous for all $m$ and $d$. He did not use the term $m$-covering number for $\gamma_m(K)$ and simply referred to it as the smallest positive homothety ratio. In [28], we proved a stronger result (Theorem
Further properties and some variants of \( \gamma_m(\cdot) \) are discussed in the recent papers [92, 168]. For instance, it has been shown in [92] that for any centrally symmetric \( d \)-dimensional convex polytope \( P \) with \( m \) vertices, we have

\[
\gamma_m(P) \leq \frac{d - 1}{d}.
\]

Obviously, any \( K \in K^d \) can be covered by \( 2^d \) smaller positive homothets if and only if \( \gamma_{2^d}(K) < 1 \). Zong used these ideas to propose a possible computer-based approach to attack the Covering Conjecture [171].

Recall that in a metric space, such as \( K^d \), an \( \epsilon \)-net \( \xi \) is a finite or infinite subset of \( K^d \) such that the union of closed balls of radius \( \epsilon \) centered at elements of \( \xi \) covers the whole space. Thus, if an \( \epsilon \)-net exists, any element of \( K^d \) is within Banach–Mazur distance \( \epsilon \) of some element of the net. The key idea of the procedure proposed by Zong is the construction of a finite \( \epsilon \)-net of \( K^d \) whose elements are convex polytopes, for every real number \( \epsilon > 1 \) and positive integer \( d \). Here we describe the construction briefly.

We first take an affine image of a \( d \)-dimensional convex body \( K \) that is sandwiched between the unit ball \( B^d \) centered at the origin and the ball \( dB^d \) with radius \( d \). Such an image always exists by John’s Ellipsoid Theorem [153, page 588]. Then we take a covering \( \{C_1, \ldots, C_m\} \) of the boundary of \( dB^d \) with spherical caps \( C_i \) as shown in Figure 2.5. The centers of the caps \( C_i \) are joined to the origin by lines \( \{L_1, \ldots, L_m\} \) and a large number of equidistant points are taken on the lines \( L_i \). We denote by \( p_i \) the point lying in \( K \cap L_i \) that is farthest from the origin. Then the convex hull \( P = \text{conv}\{p_i\} \) is the required element of our \( \epsilon \)-net. Zong [171] showed that by taking \( m \) large enough and increasing the number of points on \( L_i \) we can ensure \( d_{BM}(K, P) \leq \epsilon \).

He then notes that if we manage to construct a finite \( \epsilon \)-net \( \xi = \{P_1 : i = 1, \ldots, j\} \) of \( K^d \), satisfying \( \gamma_{2^d}(P_i) \leq c_d \) for some \( c_d < 1 \) and sufficiently small \( \epsilon \), then \( \gamma_{2^d}(K) < 1 \) would hold.
for every $K \in \mathcal{K}^d$. This would imply that the Covering Conjecture is true in dimension $d$.

The following is a four step approach suggested by Zong [171].

**Zong’s Program:**

1. For a given dimension such as $d = 3$, investigate (with the assistance of a computer) $\gamma_{2d}(K)$ for some particular convex bodies $K$ and choose a candidate constant $c_d$.

2. Choose a suitable $\epsilon$.

3. Construct an $\epsilon$-net $\xi$ of sufficiently small cardinality.

4. Check (with the assistance of a computer) that the minimal $\gamma_{2d}$-value over all elements of $\xi$ is bounded above by $c_d$.

Indeed this approach appears to be promising and, to the author’s knowledge, is a first attempt at a computer-based resolution of the Covering Conjecture. However, Zong’s program
is not without its pitfalls. For one, it would take an extensive computational experiment to come up with a good candidate constant $c_d$. Secondly, Zong’s $\epsilon$-net construction leads to a net with an exponentially large number of elements. In fact, using Böröczky and Wintsche’s estimate [51] on the number of caps in a spherical cap covering, Zong [171] showed that

$$|\xi| \leq \left\lfloor \frac{7d}{\ln \epsilon} \right\rfloor^{c_1 14^{d} 2^{d+3} (\ln \epsilon)^{-d}} \left( \ln \epsilon \right)^{-d},$$

where $c$ is an absolute constant. Since Zong’s construction does not provide much room for improving the above estimate, better constructions are needed to reduce the size of $\xi$, while at the same time keeping $\epsilon$ sufficiently small.

**Problem 2.4.1.** Develop a computationally efficient procedure for constructing $\epsilon$-nets of $\mathcal{K}^d$ of small cardinality.

Addressing the above problem would be a critical first step in implementing Zong’s program. Wu [168] (also see [92]) has recently proposed two variants of $\gamma_m(\cdot)$ that can be used in Zong’s program instead. However, the challenges and implementation issues remain the same.
Chapter 3

The covering index of convex bodies

Covering a convex body by its homothets is a classical notion in discrete geometry that has resulted in a number of interesting and long-standing problems. Swanepoel introduced the covering parameter of a convex body as a means of quantifying its covering properties. In this chapter, we introduce two relatives of the covering parameter called the covering index and the weak covering index, which upper bound well-studied quantities like the illumination number, the illumination parameter and the covering parameter of a convex body. Intuitively, the two indices measure how well a convex body can be covered by a relatively small number of homothets having the same relatively small homothety ratio. We show that the covering index is a lower semicontinuous functional on the Banach-Mazur space of convex bodies. We further show that the affine $d$-cubes minimize the covering index in any dimension $d$, while circular disks maximize it in the plane. Furthermore, the covering index satisfies a nice compatibility with the operations of direct vector sum and vector sum. In fact, we obtain an exact formula for the covering index of a direct vector sum of convex bodies that works in infinitely many instances. This, together with a minimization property, can be used to determine the covering index of infinitely many convex bodies. As the name suggests, the weak covering index loses some of the important properties of the covering index. Finally, we obtain upper bounds on the covering and weak covering index. Most of the results presented
in this chapter appear in [28].

3.1 More on $\gamma_m(\cdot)$

Before defining the covering index, we prove the following result [28], which strengthens Zong’s result [171] on the uniform continuity of $\gamma_m(\cdot)$.

**Proposition 3.1.1.** For any $K, L \in \mathcal{K}^d$,

$$\gamma_m(K) \leq d_{BM}(K, L)\gamma_m(L)$$

(3.1)

holds and so $\gamma_m$ is Lipschitz continuous on $\mathcal{K}^d$ with $\frac{d^2 - 1}{2 \ln d}$ as a Lipschitz constant and

$$|\gamma_m(K) - \gamma_m(L)| \leq d_{BM}(K, L) - 1 \leq \frac{d^2 - 1}{2 \ln d} \ln (d_{BM}(K, L)).$$

**Proof.** Let $\delta > 1$ be such that $d_{BM}(K, L) < \delta$. Now let $a \in K$, $b \in L$ and the invertible linear operator $T : \mathbb{E}^d \to \mathbb{E}^d$ satisfy $L - b \subseteq T(K - a) \subseteq \delta(L - b)$. Moreover, let $\{\lambda L + x_i : x_i \in \mathbb{E}^d, i = 1, \ldots, m\}$ be a homothetic cover of $L$, having $m$ homothets with homothety ratio $\lambda > 0$. Then

$$T(K - a) \subseteq \delta(L - b) \subseteq \delta \left( \bigcup_{i=1}^{m} (\lambda L + x_i - b) \right) = \delta \left( \bigcup_{i=1}^{m} (\lambda(L - b) + x_i + (\lambda - 1)b) \right)$$

$$\subseteq \delta \left( \bigcup_{i=1}^{m} (\lambda T(K - a) + x_i + (\lambda - 1)b) \right) = \bigcup_{i=1}^{m} (\delta\lambda T(K - a) + \delta x_i + \delta(\lambda - 1)b),$$

which implies that there is a homothetic cover of $T(K - a)$ having $m$ homothets with homothety ratio $\delta\lambda$. Hence there is a homothetic cover of $K$ having $m$ homothets with homothety ratio $\delta\lambda$. This implies that $\gamma_m(K) \leq \delta\gamma_m(L)$. Therefore, by taking $\inf \delta = d_{BM}(K, L)$, we get $\gamma_m(K) \leq d_{BM}(K, L)\gamma_m(L)$. 

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On the other hand, $\gamma_m(K) \leq 1$, $\gamma_m(L) \leq 1$ and (3.1) imply in a straightforward way that

$$|\gamma_m(K) - \gamma_m(L)| \leq d_{BM}(K, L) - 1.$$ 

If $d_{BM}(K, L) = 1$, we have nothing further to prove. Otherwise, recall John’s Ellipsoid Theorem ([153], page 588) implying $1 \leq d_{BM}(K, L) \leq d^2$. Thus, using the concavity of $\ln(\cdot)$, one obtains

$$\frac{2\ln d}{d^2-1} \leq \frac{\ln(d_{BM}(K, L))}{d_{BM}(K, L) - 1}.$$ 

Proposition 3.1.1 has some interesting consequences. For instance, it can be used to prove the following statement that settles Problem 6, posed by Brass, Moser and Pach, in Section 3.2 of [56]. We note that this statement was first proved in [139] via showing the upper semicontinuity of $\gamma_{H_d} : K^d \to \mathbb{R}$. Since we use the continuity of $\gamma_m$, our proof is simpler.

**Corollary 3.1.2** (Problem 6, Section 3.2 [56]). Let $H_d$ denote the smallest number $h$ for which every $d$-dimensional convex body can be covered by $h$ smaller positive homothetic copies of itself. Let $\overline{H}_d$ be the smallest $h$ for which there exists a positive $\lambda_d < 1$ such that every $d$-dimensional convex body can be covered by at most $h$ of its homothetic copies with homothety ratio at most $\lambda_d$. Then $H_d = \overline{H}_d$ for every $d$.

**Proof.** Clearly, $H_d \leq \overline{H}_d$. On the other hand, as the space $K^d$ is compact under the Banach-Mazur metric [124, 171], therefore $\gamma_{H_d}(K^d) \subseteq (0, 1)$ is compact as well. Thus, there is a constant $c < 1$ such that $\gamma_{H_d}(K) \leq c$ for any $K \in K^d$. As a result, we get that $\overline{H}_d \leq H_d$, finishing the proof of $\overline{H}_d = H_d$. 

Another consequence is the following.

**Corollary 3.1.3.** If $K \in K^d$ and $d_{BM}(K, C^d) < 2$, then $I(K) \leq 2^d$.

We use Proposition 3.1.1 in the sequel to study the continuity of the covering index.
### 3.2 The covering index

We now present the formal definition of the covering index.

**Definition 3.2.1.** Let $K$ be a $d$-dimensional convex body. We define the *covering index* of $K$ as

$$\text{coin}(K) = \inf \left\{ \frac{m}{1 - \gamma_m(K)} : \gamma_m(K) \leq 1/2, m \in \mathbb{N} \right\}.$$  

Intuitively, $\text{coin}(K)$ measures how $K$ can be covered by a relatively small number of positive homothets all corresponding to the same relatively small homothety ratio. We note that $\text{coin}(K)$ is an affine invariant quantity assigned to $K$, i.e., if $A : \mathbb{E}^d \rightarrow \mathbb{E}^d$ is an invertible linear map then $\text{coin}(A(K)) = \text{coin}(K)$.

We have the following relationship.

**Proposition 3.2.2.** For any $o$-symmetric $d$-dimensional convex body $K$,

$$I(K) \leq \text{ill}(K) \leq 2C(K) \leq 2 \text{coin}(K),$$

and in general for $K \in K^d$,

$$I(K) \leq C(K) \leq \text{coin}(K).$$

Proposition 3.2.2 follows immediately from the definition of coin, the relation (2.5) and the observation

$$\begin{align*}
\text{coin}(K) &= \inf \left\{ \frac{m}{1 - \gamma_m(K)} : \gamma_m(K) \leq 1/2, m \in \mathbb{N} \right\} \\
&= \inf \left\{ \frac{m}{1 - \lambda} : K \subseteq \bigcup_{i=1}^{m} (\lambda K + t_i), 0 < \lambda \leq 1/2, t_i \in \mathbb{E}^d, m \in \mathbb{N} \right\} \\
&\geq C(K).
\end{align*}$$

We remark that the inequality $\text{ill}(K) \leq 2 \text{coin}(K)$ can also be derived directly by suitably modifying the proof of Proposition 1 of Swanepoel [159].
3.2.1 Why $\gamma_m(K) \leq 1/2$?

The reader may be a bit surprised to see the restriction $\gamma_m(K) \leq 1/2$ in the definition of the covering index. There are several reasons behind this choice, which we explain below.

Given convex bodies $K, L \in \mathcal{K}^d$, the covering number of $K$ by $L$ is denoted by $N(K, L)$ and is defined as the minimum number of translates of $L$ needed to cover $K$. Among covering problems, the problem of covering the $d$-dimensional ball by smaller positive homothets has generated a lot of interest. One question that has been asked repeatedly is: what is the value of $N(B^d, \lambda B^d)$ [150, 165]? In particular, the case $\lambda = 1/2$ has attracted special attention. Verger-Gaugry [165] showed that

$$N\left(B^d, \frac{1}{2} B^d\right) = O(2^d d^{3/2} \ln d).$$

Now one immediate consequence of restricting $\gamma_m(K)$ in the definition of coin($K$) is that for any $K \in \mathcal{K}^d$,

$$N\left(K, \frac{1}{2} K\right) \leq \text{coin}(K) \leq 2N\left(K, \frac{1}{2} K\right),$$

that is, $\text{coin}(K) = \Theta(N\left(K, \frac{1}{2} K\right))$. Therefore, $\text{coin}(B^d)$ (resp. $\text{coin}(K)$) can be used to estimate $N\left(B^d, \frac{1}{2} B^d\right)$ (resp. $N\left(K, \frac{1}{2} K\right)$) and vice versa.

However, there are other more compelling reasons for choosing $1/2$ as the threshold. To understand these better, we define

$$f_m(K) = \begin{cases} 
\frac{m}{1 - \gamma_m(K)}, & \text{if } 0 < \gamma_m(K) \leq \frac{1}{2}, \\
+\infty, & \text{if } \frac{1}{2} < \gamma_m(K) \leq 1.
\end{cases}$$

Thus, $\text{coin}(K) = \inf \{f_m(K) : m \in \mathbb{N}\}$. Later, in Theorem 3.3.2, we show that for any $K, L \in \mathcal{K}^d$ and $m \in \mathbb{N}$ such that $\gamma_m(K) \leq 1/2$ and $\gamma_m(L) \leq 1/2$,

$$f_m(K) \leq d_{BM}(K, L)f_m(L),$$

(3.3)
and
\[ f_m(K) \geq \frac{d_{BM}(K, L)}{2d_{BM}(K, L) - 1} f_m(L), \]  
(3.4)

establishing a strong connection with the Banach-Mazur distance of convex bodies. The proofs of relations (3.3) and (3.4) make extensive use of homothety ratios being less than or equal to one-half. This shows that the ‘half constraint’ in the definition of the covering index results in a quantity with potentially nicer properties. In particular, relation (3.3) is important as for each \( m \), it implies Lipschitz continuity of \( f_m \) on the subspace

\[ \mathcal{K}^d_m := \{ K \in \mathcal{K}^d : \gamma_m(K) \leq 1/2 \}, \]  
(3.5)

which in turn leads to the continuity properties of coin discussed in Section 3.3. We remark that from the proof of Theorem 3.5.1, \( \mathcal{K}^d_m \neq \emptyset \) if and only if \( m \geq 2^d \).

In Section 3.6, we demonstrate what happens if we remove the restriction \( \gamma_m(K) \leq 1/2 \) from the definition of the covering index. The resulting quantity, which we call the weak covering index, loses some important properties satisfied by the covering index.

### 3.3 Continuity of \( f_m(\cdot) \) and \( \text{coin}(\cdot) \)

In this section, we establish some important properties of coin. The first observation, though trivial, helps in computing the exact values and upper estimates of coin for several convex bodies.

**Lemma 3.3.1** (Minimization lemma). *For any \( d \)-dimensional convex body \( K \), the inequality \( f_\ell(K) > f_m(K) \) implies \( m < f_\ell(K) \).*

This shows that the covering index of any convex body can be obtained by calculating a finite minimum, rather than the infimum of an infinite set. In particular, if \( f_\ell(K) < \infty \) for some \( \ell \), then \( \text{coin}(K) = \min \{ f_m(K) : m < f_\ell(K) \} \).
The next result summarizes what we know about the continuity of $f_m$ and coin. Note that the restriction $\gamma_m(K) \leq 1/2$ plays a key role throughout the proof. We remark that without this constraint (or a constraint of the form $\gamma_m(K) \leq r$, where $0 < r \leq 1/2$), the proof of Theorem 3.3.2 would not hold.

**Theorem 3.3.2** (Continuity). Let $d$ be any positive integer.

(i) For any $K, L \in \mathcal{K}^d_m$, the relations (3.3) and (3.4) hold. Moreover, equality holds in (3.3) if and only if $d_{BM}(K, L) = 1$, i.e., $L$ is an affine image of $K$ and equality in (3.4) holds if and only if either $d_{BM}(K, L) = 1$ or $d_{BM}(K, L) > 1$ with

$$\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K, L)} = \frac{1}{2d_{BM}(K, L)}.$$

(ii) The functional $f_m : \mathcal{K}^d_m \rightarrow \mathbb{R}$ is Lipschitz continuous with $\frac{d^2 - 1}{2\ln d}$ as a Lipschitz constant and

$$|f_m(K) - f_m(L)| \leq d_{BM}(K, L) - 1 \leq \frac{d^2 - 1}{2\ln d} \ln(d_{BM}(K, L)),$$

for all $K, L \in \mathcal{K}^d_m$. On the other hand, $f_m : \mathcal{K}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, for all $d$ and $m$.

(iii) Define $I_K = \{i : \gamma_i(K) \leq 1/2\} = \{i : K \in \mathcal{K}^d_i\}$, for any $d$-dimensional convex body $K$. If $I_L \subseteq I_K$, for some $K, L \in \mathcal{K}^d$, then

$$\text{coin}(K) \leq \frac{2d_{BM}(K, L) - 1}{d_{BM}(K, L)} \text{coin}(L) \leq d_{BM}(K, L) \text{coin}(L). \quad (3.6)$$

(iv) The functional coin : $\mathcal{K}^d \rightarrow \mathbb{R}$ is lower semicontinuous for all $d$.

(v) Define

$$\mathcal{K}^{d*} := \{K \in \mathcal{K}^d : \gamma_m(K) \neq 1/2, m \in \mathbb{N}\}.$$

Then the functional coin : $\mathcal{K}^{d*} \rightarrow \mathbb{R}$ is continuous for all $d$.

**Proof.** (i) To prove (3.3), let $K, L \in \mathcal{K}^d_m$. If $\gamma_m(K) \leq \gamma_m(L)$, then $f_m(K) \leq f_m(L) \leq \cdots$
$d_{BM}(K, L)f_m(L)$, with equality if and only if $d_{BM}(K, L) = 1$. Therefore, we can assume without loss of generality that $\gamma_m(K) > \gamma_m(L)$. Note that this, together with $\gamma_m(K) \leq 1/2$ and $\gamma_m(L) \leq 1/2$, implies

$$\gamma_m(K) - (\gamma_m(K))^2 > \gamma_m(L) - (\gamma_m(L))^2.$$ 

(3.7)

Thus, by using (3.1),

$$\frac{f_m(K)}{f_m(L)} = \frac{1 - \gamma_m(L)}{1 - \gamma_m(K)} < \frac{\gamma_m(K)}{\gamma_m(L)} \leq d_{BM}(K, L),$$

which gives (3.3). In addition, equality never holds in this case. Thus, equality in (3.3) holds if and only if $d_{BM}(K, L) = 1$.

Now to prove (3.4), we again use (3.1).

$$f_m(K) = \frac{m}{1 - \gamma_m(K)} \geq \frac{m}{1 - \gamma_m(L)} \frac{d_{BM}(K, L)(1 - \gamma_m(L))}{d_{BM}(K, L) - \gamma_m(L)} f_m(L),$$

with equality if and only if $\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K, L)}$.

Since $\gamma_m(L) \leq 1/2$,

$$\frac{1 - \gamma_m(L)}{d_{BM}(K, L) - \gamma_m(L)} \geq \frac{1}{2d_{BM}(K, L) - 1},$$

with equality if and only if either $d_{BM}(K, L) = 1$ or $d_{BM}(K, L) > 1$ with $\gamma_m(L) = 1/2$. Thus, (3.4) is satisfied and equality holds if and only if either $d_{BM}(K, L) = 1$ or $d_{BM}(K, L) > 1$, with $\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K, L)} = \frac{1}{2d_{BM}(K, L)}$.

(ii) The continuity on $\mathcal{K}^d_m$ is immediate, since $\gamma_m$ is continuous on $\mathcal{K}^d$ for all $d$ and $m$ [171]. The Lipschitz continuity follows from (3.3) in the same way as in Proposition 3.1.1.

For the lower semicontinuity on $\mathcal{K}^d$, we consider two cases.

**Case 1:** $f_m(K) = \frac{m}{1 - \gamma_m(K)}$, with $0 < \gamma_m(K) \leq \frac{1}{2}$. 


We need to show that for every $\epsilon > 0$, there exists $\delta > 0$, such that $f_m(K') \geq f_m(K) - \epsilon$, for all $K'$ with $1 \leq d_{BM}(K, K') \leq 1 + \delta$. Our proof of this claim is indirect.

Assume that there exist $\epsilon_0 > 0$, $\delta_1 > \delta_2 > \cdots > \delta_n > \cdots > 0$ with $\lim_{n \to +\infty} \delta_n = 0$, and $K_1, K_2, \ldots, K_n, \ldots \in \mathcal{K}^d$ such that $f_m(K_n) < f_m(K) - \epsilon_0$, where $1 \leq d_{BM}(K, K_n) \leq 1 + \delta_n$, $n = 1, 2, \ldots$. Here

\[ f_m(K_n) = \frac{m}{1 - \gamma_m(K_n)} < \frac{m}{1 - \gamma_m(K)} - \epsilon_0 = f_m(K) - \epsilon_0, \]

implying that

\[ \gamma_m(K) > 1 - \frac{m}{1 - \gamma_m(K)} - \epsilon_0 > \gamma_m(K_n) > 0. \quad (3.8) \]

As $\lim_{n \to +\infty} d_{BM}(K, K_n) = 1$ and $\gamma_m : \mathcal{K}^d \to \mathbb{R}$ is continuous, therefore, $\lim_{n \to +\infty} \gamma_m(K_n) = \gamma_m(K)$, which together with (3.8) implies $\gamma_m(K) > \gamma_m(K)$, a contradiction.

Case 2: $f_m(K) = +\infty$, with $\frac{1}{2} < \gamma_m(K) \leq 1$.

Here we need to show that for any $K_1, K_2, \ldots, K_n, \ldots \in \mathcal{K}^d$ with $\lim_{n \to +\infty} d_{BM}(K, K_n) = 1$ we have that $\lim_{n \to +\infty} f_m(K_n) = +\infty$. Again, we show this via an indirect argument. First, recall that if $f_m(K_n) < +\infty$, then $m < f_m(K_n) = \frac{m}{1 - \gamma_m(K_n)} \leq 2m$ with $0 < \gamma_m(K_n) \leq \frac{1}{2}$. Second, assume that for a subsequence $K_{i_1}, K_{i_2}, \ldots, K_{i_n}, \ldots \in \mathcal{K}^d$ with $\lim_{n \to +\infty} d_{BM}(K, K_{i_n}) = 1$ we have

\[ \lim_{n \to +\infty} f_m(K_{i_n}) = \lim_{n \to +\infty} \frac{m}{1 - \gamma_m(K_{i_n})} = \frac{m}{1 - \gamma_m(K)} \leq 2m. \]

(Here, we have once again used the continuity of $\gamma_m : \mathcal{K}^d \to \mathbb{R}$.) Thus, $\gamma_m(K) \leq \frac{1}{2}$ implying that $f_m(K) < +\infty$, a contradiction.

(iii) Note that $\text{coin}(K) = \inf \{ f_m(K) : m \in I_K \}$. The result then follows from (3.3), (3.4), and the fact that $I_L \subseteq I_K$.

(iv) Let $K \in \mathcal{K}^d$ and $h = 2^{d+1} \left( \left( \frac{2d}{d} \right)^{\frac{d}{2}} - \frac{1}{2} \right)^d d(\ln d + \ln \ln d + 5)$. From the proof of Lemma 3.3.1 and Corollary 3.7.3, $\text{coin}(K) = \min \{ f_m(K) : m \leq h \}$. In fact, since it is volumetrically impossible to cover a convex body by fewer than $2^d$ of its positive homothets of homothety.
ratio less than or equal to 1/2, \(\text{coin}(\mathbf{K}) = \min \{ f_m(\mathbf{K}) : 2^d \leq m \leq h \} \). Thus, \(\text{coin} : \mathcal{K}^d \rightarrow \mathbb{R} \)

is the pointwise minimum of finitely many lower semicontinuous functions \(f_m : \mathcal{K}^d \rightarrow \mathbb{R} \cup \{+\infty\}, 2^d \leq m \leq h \), defined on the metric space \(\mathcal{K}^d \). Since the minimum of finitely many lower semicontinuous functions defined on a metric space is lower semicontinuous, the result follows.

(v) It remains to establish the upper semicontinuity. Let \((\mathbf{K}_n)_{n \in \mathbb{N}}\) be a sequence in \(\mathcal{K}^{d^*}\) converging to \(\mathbf{K} \in \mathcal{K}^{d^*}\). We prove that \(\lim \sup \text{coin}(\mathbf{K}_n) \leq \text{coin}(\mathbf{K})\). It suffices to show that for sufficiently large \(n \in \mathbb{N}\), \(I_{\mathbf{K}_n} \subseteq I_{\mathbf{K}}\) as, from (iii), this would imply \(\text{coin}(\mathbf{K}_n) \leq d_{BM}(\mathbf{K}, \mathbf{K}_n) \text{coin}(\mathbf{K})\).

Let \(m \in I_{\mathbf{K}}\), that is, \(\gamma_m(\mathbf{K}) < 1/2\) as \(\mathbf{K} \in \mathcal{K}^{d^*}\). Also note that since \(\mathbf{K}_n \in \mathcal{K}^{d^*}\), either \(\gamma_m(\mathbf{K}_n) > 1/2\) or \(\gamma_m(\mathbf{K}_n) < 1/2\). Relation (3.1) now gives

\[
\gamma_m(\mathbf{K}_n) \leq d_{BM}(\mathbf{K}, \mathbf{K}_n) \gamma_m(\mathbf{K}),
\]

for any \(n \in \mathbb{N}\). By choosing \(n\) sufficiently large we can ensure that \(\gamma_m(\mathbf{K}_n) < 1/2\) and so \(m \in I_{\mathbf{K}_n}\).

We observe that \(\mathbf{B}^3 \in \mathcal{K}^{3^*}\) (cf. Remark 3.5.4), so \(\mathcal{K}^{3^*}\) is nonempty.

The lower semicontinuity of coin leads to some interesting consequences. On the one hand, it shows that there exists a \(d\)-dimensional convex body \(\mathbf{M}\) such that \(\text{coin}(\mathbf{M}) = \inf \{ \text{coin}(\mathbf{K}) : \mathbf{K} \in \mathcal{K}^d \}\), for all \(d\). Thus, there exists a minimizer of coin over all \(d\)-dimensional convex bodies, for all \(d\). On the other hand, although lower semicontinuity does not guarantee the existence of a coin-maximizer, it does show that \(\sup \{ \text{coin}(\mathbf{K}) : \mathbf{K} \in \mathcal{K}^d \} = \sup \{ \text{coin}(\mathbf{P}) : \mathbf{P} \in \mathcal{P}^d \}\), where \(\mathcal{P}^d\) denotes the set of all \(d\)-dimensional convex polytopes, which is known to be dense in \(\mathcal{K}^d\). Therefore, in trying to compute the supremum of coin one can restrict attention to the class of polytopes. This is not true for the illumination number, which is known to be upper semicontinuous (see [21], pp. 23-24) but is not lower semicontinuous.
We do not know whether coin is continuous on \( \mathcal{K}^d \) or not. The argument used to prove the upper semicontinuity of coin on \( \mathcal{K}^{d^*} \) does not seem to work in general. We therefore propose the following problem.

**Problem 3.3.3.** Either prove that coin is upper semicontinuous on \( \mathcal{K}^d \) or construct a counterexample.

It would be natural to ask whether analogues of inequalities (3.3) and (3.4) hold for coin. The answer is negative for both. One can look at the example of a circle \( B^2 \) and a square \( C^2 \).

It is well-known that \( d_{BM}(C^2, B^2) = \sqrt{2} \) and we will see in Section 3.5 that \( \text{coin}(B^2) = 14 \) and \( \text{coin}(C^2) = 8 \). But then \( \text{coin}(B^2) > \sqrt{2} \text{coin}(C^2) \) and \( \text{coin}(C^2) < \frac{\sqrt{2}}{2\sqrt{2}-1} \text{coin}(B^2) \).

### 3.4 Compatibility with vector sums

For the sake of brevity, we write \( N_{\lambda}(K) \) instead of \( N(K, \lambda K) \), for any \( d \)-dimensional convex body \( K \) and \( 0 < \lambda \leq 1 \). Clearly, \( N_1(K) = 1 \),

\[
N_{\gamma_m(K)}(K) \leq m \tag{3.9}
\]

and

\[
\gamma_{N_{\lambda}(K)}(K) \leq \lambda. \tag{3.10}
\]

Moreover, either inequality can be strict. To see that (3.9) can be strict, consider the example of an affine regular convex hexagon \( H \). Lassak [118] proved that \( \gamma_7(K) = 1/2 \) holds for any \( o \)-symmetric convex domain \( K \). Thus, \( \gamma_7(H) = 1/2 \). On the other hand, from Figure 3.1 and the monotonicity of \( \gamma_m(K) \) in \( m \) [171] it follows that \( 1/2 = \gamma_7(H) \leq \gamma_6(H) \leq 1/2 \). Thus, \( \gamma_6(H) = 1/2 \) and \( N_{\gamma_7(H)} = N_{1/2}(H) \leq 6 \).

To see that (3.10) can be strict, note that it is possible to have \( N_{\lambda_1}(K) = N_{\lambda_2}(K) \), for some \( \lambda_1 < \lambda_2 \). For instance, \( N_{1/2}(C^d) = N_{\lambda}(C^d) = 2^d \), for any \( 1/2 < \lambda < 1 \). Therefore
Figure 3.1: Covering $H$ by six homothets with homothety ratio $\frac{1}{2}$.

$\gamma_{N_\lambda(C^d)}(C^d) = \gamma_{2^d}(C^d) = \frac{1}{2} < \lambda$ for any $1/2 < \lambda < 1$. We use these ideas in the remainder of this section.

We now present some results showing that coin behaves very nicely with certain binary operations of convex bodies. The first five concern direct vector sums, and will be used extensively in computing the exact values and estimates of coin for higher dimensional convex bodies from the covering indices of lower dimensional convex bodies. To state these results, we introduce the notion of tightly covered convex bodies.

**Definition 3.4.1.** We say that a convex body $K \in K^d$ is *tightly covered* if for any $0 < \lambda < 1$, $K$ contains $N_\lambda(K)$ points no two of which belong to the same homothet of $K$ with homothety ratio $\lambda$.

For instance, $\ell \in K^1$ is tightly covered since for any $0 < \lambda < 1$, the line segment $\ell$ contains $N_\lambda(\ell) = \lfloor \lambda^{-1} \rfloor$ points, no two of which can be covered by the same homothet of the form $\lambda \ell + t$, $t \in E^1$. Later (see Corollary 3.4.5) we will see that for any $d \geq 2$, the $d$-dimensional cube $C^d$ is also tightly covered. We do not know of any other examples of tightly covered convex bodies. Could it be that, for $d \geq 2$, $d$-cubes are the only such convex bodies?
Problem 3.4.2. For some $d \geq 2$, find a tightly covered convex body $K \in \mathcal{K}^d$ other than $C^d$ or show that no such convex body exists.

We do, however, know that not all convex bodies are tightly covered as will be seen through the example of the circle $B^2$. (See the discussion right after the proof of Corollary 3.4.5.)

Theorem 3.4.3. Let $E^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ be a decomposition of $E^d$ into the direct vector sum of its linear subspaces $\mathbb{L}_i$ and let $K_i \subseteq \mathbb{L}_i$ be convex bodies such that $\text{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \ldots, n$, and

$$\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}.$$  

If some $n - 1$ of the $K_i$’s are tightly covered, then

$$\max\{\text{coin}(K_i) : 1 \leq i \leq n\} \leq \text{coin}(K_1 \oplus \cdots \oplus K_n) = \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^{n} N_{\lambda}(K_i)}{1 - \lambda} \leq \frac{\prod_{i=1}^{n} N_{1}(K_i)}{1 - \Gamma} \leq \frac{\prod_{i=1}^{n} m_i}{1 - \Gamma} < \prod_{i=1}^{n} \text{coin}(K_i),$$

where $K_1 \oplus \cdots \oplus K_n$ stands for the direct sum of the convex bodies $K_1 \subseteq \mathbb{L}_1, \ldots, K_n \subseteq \mathbb{L}_n$. Moreover, the first two upper bounds in (3.11) are tight.

Proof. First, we prove the lower bound for $\text{coin}(K_1 \oplus \cdots \oplus K_n)$. Let $P_{\mathbb{L}_i} : E^d \to \mathbb{L}_i$ denote the projection of $E^d$ onto $\mathbb{L}_i$ parallel to the linear subspace $\mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_{i-1} \oplus \mathbb{L}_{i+1} \oplus \cdots \oplus \mathbb{L}_n$, $i = 1, \ldots, n$. Let $\{\lambda K + x_j : x_j \in E^d, j = 1, \ldots, m\}$ be a homothetic covering of $K = K_1 \oplus \cdots \oplus K_n \subseteq E^d$ with homothety ratio $0 < \lambda \leq 1/2$. As $\{P_{\mathbb{L}_i}(\lambda K + x_j) = \lambda K_i + P_{\mathbb{L}_i}(x_j) : x_j \in E^d, j = 1, \ldots, m\}$ is a homothetic covering of $K_i$ with homothety ratio $\lambda$ in $\mathbb{L}_i$, $1 \leq i \leq n$, the lower bound follows.

Second, we prove the formula and the upper bounds on $\text{coin}(K_1 \oplus \cdots \oplus K_n)$.  

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Proposition 3.4.4. If some $n - 1$ of the $K_i$'s are tightly covered, then for all $0 < \lambda < 1$,

$$N_\lambda(K_1 \oplus \cdots \oplus K_n) = \prod_{i=1}^{n} N_\lambda(K_i).$$ \hfill (3.12)

Proof. Let $N_i = N_\lambda(K_i)$, $i = 1, \ldots, n$, and let $\{\lambda K_i + t_{ij} : t_{ij} \in \mathbb{L}_i, j = 1, \ldots, N_i\}$ be a homothetic covering of $K_i$ with homothety ratio $\lambda$ in $\mathbb{L}_i$, for $i = 1, \ldots, n$.

Clearly,

$$\{\lambda K_1 + t_{1j_1} \oplus \cdots \oplus (\lambda K_n + t_{nj_n}) : t_{ij} \in \mathbb{L}_i, i = 1, \ldots, n, j = 1, \ldots, N_i\}$$

$$= \{\lambda (K_1 \oplus \cdots \oplus K_n) + t_{1j_1} + \cdots + t_{nj_n} : i = 1, \ldots, n, j = 1, \ldots, N_i\}$$

is a homothetic covering of $K_1 \oplus \cdots \oplus K_n$ with homothety ratio $\lambda$ in $\mathbb{E}^d$ having cardinality $\prod_{i=1}^{n} N_i$. Thus, $N_\lambda(K_1 \oplus \cdots \oplus K_n) \leq \prod_{i=1}^{n} N_\lambda(K_i)$.

Next, let $C = \{\lambda (K_1 \oplus \cdots \oplus K_n) + t_j : t_j \in \mathbb{E}^d, j = 1, \ldots, N\}$ be a minimal cardinality homothetic covering of $K_1 \oplus \cdots \oplus K_n$ with homothety ratio $\lambda$ in $\mathbb{E}^d$. Let us assume, without loss of generality, that $K_1, \ldots, K_{n-1}$ are tightly covered. So, for $i = 1, \ldots, n - 1$ and $j_i = 1, \ldots, N_\lambda(K_i)$, there exist points $x_{ij_i} \in K_i$ such that for any fixed $i$ and $1 \leq j_i \neq j'_i \leq N_\lambda(K_i)$, $x_{ij_i}$ and $x_{ij'_i}$ cannot both be contained in a homothet of $K_i$ with homothety ratio $\lambda$. Therefore, no homothet in $C$ intersects any two of the $\prod_{i=1}^{n-1} N_\lambda(K_i)$ cross sections $x_{ij_1} + \cdots + x_{(n-1)j_{n-1}} + K_n$ of $K_1 \oplus \cdots \oplus K_n$. In order to cover each such cross section, we require at least $N_\lambda(K_n)$ homothets from $C$. Thus, $N_\lambda(K_1 \oplus \cdots \oplus K_n) = N \geq \prod_{i=1}^{n} N_\lambda(K_i)$. \hfill $\square$

Hence, for any $0 < \lambda < 1$,

$$\frac{N_\lambda(K_1 \oplus \cdots \oplus K_n)}{1 - \lambda} = \frac{\prod_{i=1}^{n} N_\lambda(K_i)}{1 - \lambda}.$$
Thus,

$$\text{coin}(K_1 \oplus \cdots \oplus K_n) = \inf_{m \in \mathbb{N}} \left\{ \frac{m}{1 - \gamma_m(K_1 \oplus \cdots \oplus K_n)} : \gamma_m(K_1 \oplus \cdots \oplus K_n) \leq \frac{1}{2} \right\}$$

$$= \inf_{\lambda \leq \frac{1}{2}} \frac{N_\lambda(K_1 \oplus \cdots \oplus K_n)}{1 - \lambda}$$

$$= \inf_{\lambda \leq \frac{1}{2}} \prod_{i=1}^{n} \frac{N_\lambda(K_i)}{1 - \lambda},$$

completing the proof of the equality appearing in (3.11).

The upper bounds in (3.11) now follow from the definition of $\Gamma$ and $m_i$, $i = 1, \ldots, n$. Moreover, the example of $d$-cubes, considered as direct vector sums of $d$ 1-dimensional line segments, shows that the first two upper bounds in (3.11) are tight (cf. Theorem 3.5.1).

We have the following immediate corollary of Proposition 3.4.4, which shows that $d$-cubes are tightly covered.

**Corollary 3.4.5.** Let $\mathbb{E}^d = L_1 \oplus \cdots \oplus L_n$ be a decomposition of $\mathbb{E}^d$ into the direct vector sum of its linear subspaces $L_i$ and let $K_i \subseteq L_i$, $i = 1, \ldots, n$, be tightly covered convex bodies. Then $K_1 \oplus \cdots \oplus K_n$ is tightly covered.

**Proof.** For any $0 < \lambda < 1$, allowing $K_n$ to be tightly covered in the proof of Proposition 3.4.4 yields $\prod_{i=1}^{n} N_\lambda(K_i) = N_\lambda(K_1 \oplus \cdots \oplus K_n)$ points in the convex body $K_1 \oplus \cdots \oplus K_n$, no two of which belong to the same homothet of $K_1 \oplus \cdots \oplus K_n$ with homothety ratio $\lambda$.

Boltyanski and Martini [49] showed that $I(K_1 \oplus \cdots \oplus K_n) \leq \prod_{j=1}^{n} I(K_j)$, but that the equality does not hold in general since $I(B^2 \oplus B^2) = 7 < 9 = (I(B^2))^2$. Thus, there exists $\lambda < 1$ such that $N_\lambda(B^2 \oplus B^2) = 7$, whereas $N_\lambda(B^2) = 3$. Hence, relation (3.12) does not hold, and by Proposition 3.4.4, $B^2$ is not tightly covered.

Although the inequality $N_\lambda(K_1 \oplus \cdots \oplus K_n) \leq \prod_{i=1}^{n} N_\lambda(K_i)$ always holds, the example of $B^2 \oplus B^2$ shows that the equality (3.12) is not satisfied in general. We have the following general result on the covering index of direct vector sums of convex bodies.
Corollary 3.4.6. Let $E^d = L_1 \oplus \cdots \oplus L_n$ be a decomposition of $E^d$ into the direct vector sum of its linear subspaces $L_i$ and let $K_i \subseteq L_i$ be convex bodies such that $\text{coin}(K_i) = f_{m_i}(K_i), \quad i = 1, \ldots, n$, and $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}$. Then

$$\max\{\text{coin}(K_i) : 1 \leq i \leq n\} \leq \text{coin}(K_1 \oplus \cdots \oplus K_n) \leq \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda} \leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1 - \Gamma} \leq \frac{\prod_{i=1}^n m_i}{1 - \Gamma} \leq \prod_{i=1}^n \text{coin}(K_i).$$

(3.13)

Moreover, the first three upper bounds in (3.13) are tight.

Let $K \subseteq E^{d-k} \subseteq E^{d-k} \oplus E^{1} \oplus \cdots \oplus E^{1} = E^d$ be a $(d-k)$-dimensional convex body and $\ell \subseteq E^{1} \subseteq E^d$ denote a line segment that can be optimally covered (in the sense of coin) by two homothets of homothety ratio $1/2$. We say that the $d$-dimensional convex body

$$K \oplus \ell \oplus \cdots \oplus \ell \subseteq E^d$$

is a (bounded) $k$-codimensional cylinder. We have seen that the covering index behaves nicely with direct vector sums. We now show that in case of 1-codimensional cylinders it behaves even more nicely.

Corollary 3.4.7. For any 1-codimensional $d$-dimensional cylinder $K \oplus \ell$, the first two upper bounds in (3.11) become equalities and

$$\text{coin}(K \oplus \ell) = 4N_{1/2}(K).$$
Proof. First note that since \( \ell \) is tightly covered, Theorem 3.4.3 is applicable. From (3.11),

\[
\text{coin}(K \oplus \ell) = \inf_{\lambda \leq \frac{1}{2}} \frac{N_\lambda(K)N_\lambda(\ell)}{1 - \lambda} = \inf_{\lambda \leq \frac{1}{2}} \frac{N_\lambda(K)\lceil \lambda^{-1} \rceil}{1 - \lambda} \\
\leq \frac{N_{1/2}(K)N_{1/2}(\ell)}{1 - \frac{1}{2}} = 4N_{1/2}(K).
\]

Suppose for some \( 0 < \lambda < \frac{1}{2} \), \( \frac{N_\lambda(K)\lceil \lambda^{-1} \rceil}{1 - \lambda} < 4N_{1/2}(K) \). Then

\[
\lceil \lambda^{-1} \rceil \frac{N_\lambda(K)}{N_{1/2}(K)} < 4(1 - \lambda),
\]

which is impossible, since, for \( 0 < \lambda < \frac{1}{2} \), \( \lceil \lambda^{-1} \rceil \geq 4(1 - \lambda) \) and \( N_\lambda(K) \geq N_{1/2}(K) \).

Thus,

\[
\text{coin}(K \oplus \ell) = 4N_{1/2}(K).
\]

In addition to the direct vector sum, coin displays a compatibility with the Minkowski sum (or simply vector sum) of convex bodies. We note that the upper bounds appearing here are the same as in Corollary 3.4.6.

Theorem 3.4.8. Let the convex body \( K \) be the vector sum of the convex bodies \( K_1, \ldots, K_n \) in \( \mathbb{E}^d \), i.e., let \( K = K_1 + \cdots + K_n \) such that \( \text{coin}(K_i) = f_{m_i}(K_i) \), \( i = 1, \ldots, n \), and \( \Gamma = \max \{ \gamma_{m_i}(K_i) : 1 \leq i \leq n \} \). Then

\[
\text{coin}(K) \leq \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda} \leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1 - \Gamma} \leq \frac{\prod_{i=1}^n m_i}{1 - \Gamma} < \prod_{i=1}^n \text{coin}(K_i). \tag{3.14}
\]

Moreover, equality in (3.14) does not hold in general.

Proof. Given homothetic coverings of \( K_i, i = 1, \ldots, n \), with homothety ratio \( 0 < \lambda \leq 1/2 \), one can construct a homothetic covering of \( K = K_1 + \cdots + K_n \) with the same homothety ratio \( \lambda \) in a natural way. The proof of the upper bounds follows along the same lines as in
Theorem 3.4.3 and Corollary 3.4.6. Furthermore, to show that equality in (3.14) does not hold in general, we consider the example of an affine regular convex hexagon \( H = \Delta^2 + (-\Delta^2) \) and the corresponding triangle \( \Delta^2 \).

Belousov [12] showed that \( \gamma_6(\Delta^2) = \frac{1}{2} \) and \( \gamma_m(\Delta^2) > \frac{1}{2} \), for \( 1 \leq m < 6 \). By Lemma 3.3.1, \( \text{coin}(\Delta^2) = \inf \{ f_m(\Delta^2) : 6 \leq m < 12 \} \leq f_6(\Delta^2) = 12 \). But Fudali [76] determined \( \gamma_m(\Delta^2) \), for \( 7 \leq m \leq 15 \), and routine calculations show that the corresponding \( f_m \)'s satisfy \( f_m(\Delta^2) > 12 \). Thus, \( \text{coin}(\Delta^2) = 12 \). Now, Figure 3.1 shows that \( H \) can be covered by 6 half-sized homothets. Thus, \( \text{coin}(H) \leq 12 = \text{coin}(\Delta^2) \).

It is, in fact, easy to show that \( \text{coin}(H) = 12 \). First, observe that any translate of \( \frac{1}{2}H \) can cover at the most one-sixth of the boundary of \( H \). Therefore, \( \gamma_m(H) > 1/2 \), for \( m = 1, \ldots, 5 \). Thus, as in the case of \( \Delta^2 \), \( \text{coin}(H) = \inf \{ f_m(H) : 6 \leq m < 12 \} \leq 12 \). If \( f_m(H) < 12 \), for some \( 7 \leq m \leq 11 \), then by definition of \( f_m(\cdot) \), \( \gamma_m(H) < \frac{12-m}{12} \), and by the definition of covering, \( m \gamma_m(H)^2 \text{vol}_2(H) \geq \text{vol}_2(H) \). Therefore, \( m \left( \frac{12-m}{12} \right)^2 > 1 \), which is impossible for \( 8 \leq m \leq 11 \). This only leaves the case \( m = 7 \), but it is known [118] that (cf. the remarks immediately following (3.10)) \( \gamma_7(H) = 1/2 \) and as a result, \( f_7(H) = 14 \). We conclude that \( \text{coin}(H) = 12 \). This kind of ‘volumetric’ argument will remain useful throughout the next section in determining covering index values for convex bodies. Also, Lemma 3.3.1 plays an important role, reducing the problem to finding the minimum of a finite set.

We now present an application of Theorem 3.4.8 to the difference body \( K - K = K + (-K) \) of a convex body \( K \). The result is quite useful for non-symmetric convex bodies. Once again, from the example of an affine regular convex hexagon and a triangle, we note that equality does not hold in general.

**Corollary 3.4.9.** If \( K \) is any \( d \)-dimensional convex body such that \( \text{coin}(K) = f_m(K) \), then

\[
\text{coin}(K - K) \leq \left( \frac{N_m(K)}{1 - \gamma_m(K)} \right)^2 \leq \frac{m^2}{1 - \gamma_m(K)} < (\text{coin}(K))^2.
\]

Moreover, equality in (3.15) does not hold in general.
Since the upper bounds given in relations (3.14) and (3.15) match the upper bounds in (3.13), it is natural to ask if the same is true for the lower bounds. However, the arguments used in the proof of Theorem 3.4.3 and Theorem 3.4.8 do not seem to settle this question.

**Problem 3.4.10.** Let $K_1, \ldots, K_n$ be $d$-dimensional convex bodies, for some $d \geq 2$. Then prove (disprove) that

$$\max\{\text{coin}(K_i) : i = 1, \ldots, n\} \leq \text{coin}(K_1 + \cdots + K_n).$$

(3.16)

If this does not hold, one can try proving the following weaker lower bound.

$$\min\{\text{coin}(K_i) : i = 1, \ldots, n\} \leq \text{coin}(K_1 + \cdots + K_n).$$

(3.17)

The example of a triangle and a hexagon considered above indicates that either lower bound, if it holds, would be tight. The conjectured relations (3.16) and (3.17) both lead to interesting consequences, which we discuss below.

If the weaker result (3.17) is satisfied, combining it with Corollary 3.4.9 would give $\text{coin}(K) \leq \text{coin}(K - K)$. This would show that for any convex body $K$, the $o$-symmetric convex body $K - K$ has a covering index at least as large as $\text{coin}(K)$. This, in turn, would imply that in computing the supremum of $\text{coin}(K)$ over all $d$-dimensional convex bodies one could restrict attention to the class of $o$-symmetric convex polytopes.

If the stronger result (3.16) holds, we would be able to say even more. It is known that any nonempty intersection of translates of $B^d$ is a Minkowski summand of $B^d$ (see [153], Theorem 3.2.5). This includes the class of all $d$-dimensional ball-polyhedra [33], which are nonempty intersections of finitely many translates of $B^d$. Result (3.16) would imply that $\text{coin}(B^d)$ upper bounds the covering indices of ball-polyhedra, or more generally of nonempty intersections of translates of $B^d$.  

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3.5 Extremal bodies

The aim of this section is to characterize the convex bodies that maximize or minimize the covering index among all $d$-dimensional convex bodies. In addition, we compute exact values and estimates of the covering index for a number of convex bodies.

Since coin is a lower semicontinuous functional defined on the compact space $K^d$, it is guaranteed to achieve its infimum over $K^d$, that is, there exists $M \in K^d$ such that $\text{coin}(M) = \inf \{ \text{coin}(K) : K \in K^d \}$. We have the following assertion about the minimizers of coin.

**Theorem 3.5.1.** Let $d$ be any positive integer and $K \in K^d$. Then $\text{coin}(C^d) = 2^{d+1} \leq \text{coin}(K)$ and thus, (affine) $d$-cubes minimize the covering index in all dimensions.

**Proof.** Clearly, $C^d$ can be covered by $2^d$ homothets of homothety ratio $1/2$, and cannot be covered by fewer homothets. Therefore, $\text{coin}(C^d) \leq f_{2^d}(C^d) = 2^{d+1}$. Let $p$ be a positive integer. If there exists a homothetic covering of $C^d$ by $m = 2^d + p$ homothets giving $f_m(C^d) < 2^{d+1}$, then

$$\gamma_m(C^d) < \frac{1}{2} - \frac{p}{2^{d+1}}.$$  

However,

$$m \text{vol}_d(\gamma_m(C^d)C^d) = m\gamma_m(C^d)^d \text{vol}_d(C^d) < (2^d + p) \left[ \frac{1}{2} - \frac{p}{2^{d+1}} \right]^d \text{vol}_d(C^d) < \text{vol}_d(C^d),$$

a contradiction, showing that $\text{coin}(C^d) = 2^{d+1}$.

Now consider an arbitrary $d$-dimensional convex body $K$. By repeating the above calculations for $K$ we see that for $m > 2^d$, $f_m(K)$ cannot be smaller than $2^{d+1}$. A similar volumetric argument shows that $K$ cannot be covered by $2^d$ homothets having homothety ratio less than $1/2$. Likewise, it is impossible to cover $K$ by fewer than $2^d$ homothets if the homothety ratio does not exceed $1/2$. Thus $\text{coin}(K) \geq 2^{d+1}$. 

It is known that $C(C^d) = 2^{d+1}$ [159]. Thus, $\text{coin}(C^d) = C(C^d)$. Do affine $d$-cubes also minimize the covering parameter? The answer is negative in general and open for
\(d = 2, 3\). An affine regular \(d\)-simplex \(\Delta^d\) can be covered by \(d + 1\) homothetic copies each with homothety ratio \(d/(d+1)\). Thus, \(C(\Delta^d) \leq (d+1)^2\), which is less than \(C(C^d)\) for \(d > 3\).

The question of which convex bodies minimize (or maximize) the covering parameter is wide open, even in the plane. Restricting the homothety ratio to not exceed half plays a crucial role in determining the optimizers of the covering index.

The case of coin-maximizers is more involved. Indeed, since we have not established the upper semicontinuity of coin, it may be the case that for some \(d\), \(\sup \{\text{coin}(K) : K \in \mathcal{K}^d\}\) is not achieved by any \(d\)-dimensional convex body. However, this is not the case for \(d = 2\).

**Theorem 3.5.2.** *If \(K\) is a convex domain then \(\text{coin}(K) \leq \text{coin}(B^2) = 14\).*

*Proof.* First, we show that \(\text{coin}(B^2) = 14\). It is rather trivial that \(\gamma_{1}(B^2) = \gamma_{2}(B^2) = 1, \gamma_{3}(B^2) = \sqrt{3}/2 = 0.866\ldots\), and \(\gamma_{4}(B^2) = 1/\sqrt{2} = 0.707\ldots\). Hence, \(f_{1}(B^2) = f_{2}(B^2) = f_{3}(B^2) = f_{4}(B^2) = +\infty\). Moreover, Bezdek [15] showed that \(\gamma_{5}(B^2) = 0.609\ldots\) and \(\gamma_{6}(B^2) = 0.555\ldots\), implying that \(f_{5}(B^2) = f_{6}(B^2) = +\infty\). On the other hand, it is easy to see that \(\gamma_{7}(B^2) = 1/2\) and therefore \(f_{7}(B^2) = 14\). Hence Lemma 3.3.1 implies that \(\text{coin}(B^2) = \min \{f_{m}(B^2) : 7 \leq m < 14\} \leq f_{7}(B^2) = 14\).

Next, recall G. Fejes Tóth’s result [73] according to which \(\gamma_{8}(B^2) = 0.445\ldots\) and \(\gamma_{9}(B^2) = 1/(1 + \sqrt{2}) = 0.414\ldots\). This implies \(f_{8}(B^2) = 14.420\ldots \geq 14\) and \(f_{9}(B^2) = 15.363\ldots \geq 14\).

We claim that \(f_{m}(B^2) > 14\), for all \(10 \leq m < 14\). Suppose for some \(10 \leq m < 14\), \(f_{m}(B^2) \leq 14\). In this case, we must have \(\gamma_{m}(B^2) \leq \frac{14-m}{14}\) and \(m \text{vol}_2(\gamma_{m}(B^2) B^2) > \text{vol}_2(B^2)\). This implies \(m \left(\frac{14-m}{14}\right)^2 > 1\). But routine calculations show that the latter inequality fails to hold for all \(10 \leq m \leq 13\). Thus \(\text{coin}(B^2) = 14\).

Levi [121] showed that any convex domain \(K\) can be covered by 7 homothets of homothety ratio 1/2. Thus, \(\text{coin}(K) \leq 14\), proving that the circle maximizes the covering index in the plane. \(\square\)

Although the question of maximizers is open in general, we can use Corollary 3.4.7
and Theorem 3.5.2 to determine the maximizer among 1-codimensional cylinders in $K^3$. In addition, we determine the covering indices of several 1-codimensional cylinders.

**Corollary 3.5.3.** We have the following:

(i) $\text{coin}(\Delta^2 \oplus \ell) = 24$.

(ii) $\text{coin}(H \oplus \ell) = 24$.

(iii) $\text{coin}(B^2 \oplus \ell) = 28$.

(iv) If $K \oplus \ell$ is a 1-codimensional cylinder in $K^3$, then $\text{coin}(K \oplus \ell) \leq 28$, that is $B^2 \oplus \ell$ maximizes $\text{coin}$ among 3-dimensional 1-codimensional cylinders.

**Proof.** The assertions (i)-(iii) follow immediately from Corollary 3.4.7 and the values of $\text{coin}(\Delta^2)$, $\text{coin}(H)$ and $\text{coin}(B^2)$ determined earlier. For (iv), recall that [121] for a convex domain $K$, $\max N_{1/2}(K) = 7$. □

So far, we have computed the covering index mostly for convex domains. Since in higher dimensions very little is known about exact values of $\gamma_m(K)$, it is a lot harder to determine exact values of coin. In some cases it is possible to derive upper bounds. For instance, we make the following observation for $d$-dimensional balls.

**Remark 3.5.4.** Recently, O’Rourke [130] raised the question of the minimum number of homothets of homothety ratio $1/2$ needed to cover $B^3$. Using spherical cap coverings, Wynn [130] showed this number to be 21. Thus $N_{1/2}(B^3) = 21$. In fact, Wynn also demonstrated that if the homothety ratio is decreased to $0.49439$, we can still cover $B^3$ by 21 homothets. Figure 3.2 illustrates such a covering. Therefore, $\text{coin}(B^3) \leq f_{21}(B^3) \leq 41.5339886473764$. Moreover, by applying Corollary 3.4.7, $\text{coin}(B^3 \oplus \ell) = 84$.

In general, Verger-Gaugry [165] showed that in any dimension $d \geq 2$ one can cover a ball of radius $1/2 < r \leq 1$ with $O((2r)^{d-1}d^{3/2} \ln d)$ balls of radius 1/2. Substituting $r = 1$ and performing the standard covering index calculations shows that $\text{coin}(B^d) = O(2^d d^{3/2} \ln d)$.

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The above remark is interesting for three different reasons. First, we observed that for $\mathbf{B}^2$, $\mathbf{C}^2$ and $\Delta^2$, the value of the covering index is associated with the homothety ratio 1/2. Theorem 3.5.4 provides us with an example, namely $\mathbf{B}^3$, where the covering index is associated with a homothety ratio strictly less than 1/2. Thus, half-sized homothets do not always correspond to the covering index values. Second, Remark 3.5.4 provides another example of a situation when inequality (3.10) is strict, as $\gamma_{N_{1/2}}(\mathbf{B}^3) = \gamma_{21}(\mathbf{B}^3) < 1/2$. Finally, since $\mathbf{B}^2$ maximizes the covering index in the plane, it can be asked if the same is true for $\mathbf{B}^d$ in higher dimensions.

**Problem 3.5.5.** For any d-dimensional convex body $\mathbf{K}$, prove or disprove that $\text{coin}(\mathbf{K}) \leq \text{coin}(\mathbf{B}^d)$ holds.

An affirmative answer to Problem 3.5.5 would considerably improve the known general
(Rogers-type) upper bound on the illumination number. Recall (2.2) and (2.3). We note that if $B^d$ maximizes the covering index, then the general asymptotic bound in (2.2) would improve to within a factor $\sqrt{d}$ of the bound (2.3) in the $o$-symmetric case.

Table 3.1: Known values (or estimates) of coin. The table can be extended indefinitely by including values (or estimates) of $\text{coin}(K \oplus L)$ and by including upper bounds on $\text{coin}(K + L)$, for any convex bodies $K$ and $L$ appearing in the table.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$m$</th>
<th>$\gamma_m(K)$</th>
<th>$\text{coin}(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>2</td>
<td>1/2</td>
<td>4</td>
</tr>
<tr>
<td>$H$</td>
<td>6</td>
<td>1/2</td>
<td>12</td>
</tr>
<tr>
<td>$\Delta^2$</td>
<td>6</td>
<td>1/2</td>
<td>12</td>
</tr>
<tr>
<td>$B^2$</td>
<td>7</td>
<td>1/2</td>
<td>14</td>
</tr>
<tr>
<td>$B^r$</td>
<td>$\geq 21$</td>
<td>$\leq 0.49439$</td>
<td>$\leq 41.53398 \ldots$</td>
</tr>
<tr>
<td>$B^d$</td>
<td>$O(2^d d^{3/2} \ln d)$</td>
<td>$\leq 1/2$</td>
<td>$O(2^d d^{3/2} \ln d)$</td>
</tr>
<tr>
<td>$C^d$</td>
<td>$2^d$</td>
<td>1/2</td>
<td>$2^{d+1}$</td>
</tr>
<tr>
<td>$H \oplus \ell$</td>
<td>12</td>
<td>1/2</td>
<td>24</td>
</tr>
<tr>
<td>$\Delta^2 \oplus \ell$</td>
<td>12</td>
<td>1/2</td>
<td>24</td>
</tr>
<tr>
<td>$B^2 \oplus \ell$</td>
<td>14</td>
<td>1/2</td>
<td>28</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

We conclude by listing some of the known values (or estimates) of the covering index. We remark that Table 3.1 can be continued indefinitely by using the operations of direct vector addition and the Minkowski addition, resulting in infinitely many convex bodies for which we know exact values of coin, and infinitely many convex bodies for which we can estimate coin.

### 3.6 The weak covering index

In this section, we introduce a variant of the covering index, which we call the *weak covering index*.

**Definition 3.6.1.** Let $K$ be a $d$-dimensional convex body. We define the *weak covering index*...
index of $K$ as

$$
\text{coin}_w(K) = \inf \left\{ \frac{m}{1 - \gamma_m(K)} : \gamma_m(K) < 1, m \in \mathbb{N} \right\}.
$$

Let us define

$$
g_m(K) = \begin{cases} 
\frac{m}{1 - \gamma_m(K)}, & \text{if } 0 < \gamma_m(K) < 1, \\
+\infty, & \text{if } \gamma_m(K) = 1.
\end{cases}
$$

Then $\text{coin}_w(K) = \inf \{g_m(K) : m \in \mathbb{N}\}$.

Some properties of the weak covering index naturally mirror the corresponding properties of the covering index. These include Proposition 3.2.2, Lemma 3.3.1, Theorem 3.4.3, Corollary 3.4.6 and Theorem 3.4.8. The corresponding statements can be obtained by replacing coin with $\text{coin}_w$ and $f_m$ by $g_m$ throughout.

However, no suitable analogue of Corollary 3.4.7 exists for $\text{coin}_w$. As a result, we can only estimate the weak covering index of 1-codimensional cylinders in Table 3.2. Also the discussed aspects of continuity of the covering index (Section 3.3) seem to be lost for the weak covering index. More importantly, the problem of finding the maximizers and minimizers of $\text{coin}_w$ seems a lot harder than the corresponding problem for coin. We only know a minimizer for $d = 2$.

Table 3.2: Known values (or estimates) of $\text{coin}_w(\cdot)$ together with the corresponding $m$ and $\gamma_m(\cdot)$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$m$</th>
<th>$\gamma_m(K)$</th>
<th>$\text{coin}_w(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>2</td>
<td>1/2</td>
<td>4</td>
</tr>
<tr>
<td>$H$</td>
<td>6</td>
<td>1/2</td>
<td>$\leq 12$</td>
</tr>
<tr>
<td>$\Delta^2$</td>
<td>3</td>
<td>2/3</td>
<td>9</td>
</tr>
<tr>
<td>$B^2$</td>
<td>5</td>
<td>0.609</td>
<td>12.800</td>
</tr>
<tr>
<td>$C^d$</td>
<td>$2^d$</td>
<td>1/2</td>
<td>$2^{d+1}$</td>
</tr>
<tr>
<td>$\Delta^d$</td>
<td>$\geq d + 1$</td>
<td>$\leq \frac{d}{d+1}$</td>
<td>$\leq (d + 1)^2$</td>
</tr>
<tr>
<td>$H \oplus \ell$</td>
<td>12</td>
<td>1/2</td>
<td>$\leq 24$</td>
</tr>
<tr>
<td>$\Delta^2 \oplus \ell$</td>
<td>$\geq 6$</td>
<td>$\leq 2/3$</td>
<td>$\leq 18$</td>
</tr>
<tr>
<td>$B^2 \oplus \ell$</td>
<td>$\geq 10$</td>
<td>$\leq 0.609$</td>
<td>$\leq 25.60$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Theorem 3.6.2. Let $K \in \mathcal{K}^2$. Then $\text{coin}_w(K) \geq \text{coin}_w(C^2) = 8$. Thus the (affine) square minimizes the weak covering index in the plane.

**Proof.** If $K$ is such that $\text{coin}_w(K) = g_m(K) < 8$, then from the proof of Theorem 3.5.1, $m < 4$. Since any convex body in $\mathcal{K}^2$ requires at least 3 smaller positive homothets to cover it, we only need to consider the case $m = 3$. But Belousov [12] showed that

$$\min_{K \in \mathcal{K}^2} \gamma_3(K) = \frac{2}{3},$$

and so $g_3(K) \geq 9 > \text{coin}_w(C^2)$, a contradiction. \qed

It is worth noting that for $d \geq 3$, the simplex $\Delta^d$ gives a smaller value $(\leq (d + 1)^2)$ of $\text{coin}_w$ than the $d$-cube $C^d$. Thus $\text{coin}_w$ has different minimizers in different dimensions.

### 3.7 Bounds on the covering indices

In this section, we obtain upper bounds on the covering and the weak covering index in the spirit of Rogers’ bounds on covering numbers. The main ingredients include Rogers’ estimate [149] of the infimum $\theta(K)$ of the covering density of $E^d$ by translates of the convex body $K$, namely, for $d \geq 2$,

$$\theta(K) \leq d(\ln d + \ln \ln d + 5),$$

the Rogers-Shephard inequality [151]

$$\text{vol}_d(K - K) \leq \left( \frac{2d}{d} \right) \text{vol}_d(K)$$

on the volume of the difference body, and a well-known result of Rogers and Zong [152], which states that for $d$-dimensional convex bodies $K$ and $L$, $d \geq 2$,

$$N(K, L) \leq \frac{\text{vol}_d(K - L)}{\text{vol}_d(L)} \theta(L), \quad (3.18)$$

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with $K - L = K + (-L)$. In addition, we make use of the Brunn–Minkowski inequality

$$(\text{vol}_d(K + L))^{1/d} \geq (\text{vol}_d(K))^{1/d} + (\text{vol}_d(L))^{1/d},$$

for convex bodies $K, L \subseteq \mathbb{E}^d$. The above inequalities yield the well-known upper bounds (2.2) and (2.3) on the illumination number as discussed in Chapter 2.

**Theorem 3.7.1.** Given $K \in \mathcal{K}^d$, $d \geq 2$ and a real number $0 < \lambda < 1$, we have

$$\text{coin}_w(K) \leq \frac{N_\lambda(K)}{1 - \lambda} \leq \begin{cases} \frac{(1 + \lambda)^d}{\lambda^d(1 - \lambda)}d(\ln d + \ln \ln d + 5), & \text{if } K \text{ is o-symmetric}, \\ \frac{1}{\lambda^d(1 - \lambda)} \left( \left( \frac{2d}{d} \right)^{1/d} - 1 + \lambda \right)^d d(\ln d + \ln \ln d + 5), & \text{otherwise}. \end{cases} \tag{3.19}$$

**Proof.** Consider a minimal cardinality covering of $K$ by homothets $\lambda K + t_i$, for some $t_i \in \mathbb{E}^d$, $i = 1, \ldots, N_\lambda(K)$. By (3.18), we have

$$N_\lambda(K) \leq \frac{\text{vol}_d(K - \lambda K)}{\text{vol}_d(\lambda K)} \theta(\lambda K) = \frac{\text{vol}_d(K - \lambda K)}{\text{vol}_d(\lambda K)} \frac{\text{vol}_d(\lambda K)}{\text{vol}_d(K)} \theta(K) \leq \frac{\text{vol}_d(K - \lambda K)}{\text{vol}_d(\lambda K)} d(\ln d + \ln \ln d + 5). \tag{3.20}$$

If $K$ is o-symmetric, then $\text{vol}_d(K - \lambda K) = \text{vol}_d((1 + \lambda)K) = \frac{(1 + \lambda)^d}{\lambda^d} \text{vol}_d(\lambda K)$ and so, (3.20) implies

$$N_\lambda(K) \leq \frac{(1 + \lambda)^d}{\lambda^d} d(\ln d + \ln \ln d + 5).$$

In the general case, applying the Brunn-Minkowski inequality gives

$$\lambda^{-1} \text{vol}_d(K - K)^{1/d} = \text{vol}_d\left((\lambda^{-1}K - K) + (\lambda^{-1} - 1)K\right)^{1/d} \geq \text{vol}_d(\lambda^{-1}K - K)^{1/d} + \text{vol}_d((\lambda^{-1} - 1)K)^{1/d} = \lambda^{-1} \text{vol}_d(K - K)^{1/d} + (\lambda^{-1} - 1) \text{vol}_d(K)^{1/d},$$
which gives

\[ \text{vol}_d (K - \lambda K)^{1/d} \leq \text{vol}_d (K - K)^{1/d} - (\lambda^{-1} - 1) \lambda \text{vol}_d (K)^{1/d}. \]

By the Rogers-Shephard inequality, we have

\[ \text{vol}_d (K - \lambda K)^{1/d} \leq \left( \frac{2d}{d} \right)^{1/d} \text{vol}_d (K)^{1/d} - (1 - \lambda) \text{vol}_d (K)^{1/d} \]

\[ = \lambda^{-1} \left( \left( \frac{2d}{d} \right)^{1/d} - 1 + \lambda \right) \text{vol}_d (\lambda K)^{1/d}. \]

Substituting for \( \text{vol}_d (K - \lambda K) \) in (3.20) gives

\[ N_\lambda (K) \leq \lambda^{-d} \left( \left( \frac{2d}{d} \right)^{1/d} - 1 + \lambda \right)^d d(\ln d + \ln \ln d + 5). \]

Finally, note that clearly \( \text{coin}_w(K) \leq \frac{N_\lambda(K)}{1 - \lambda} \). The upper bounds in (3.19) follow.

For \( \lambda = \frac{d}{d+1} \), Theorem 3.7.1 gives the following upper bounds on the weak covering index.

**Corollary 3.7.2.** Let \( K \in \mathcal{K}^d \), \( d \geq 2 \). Then

\[ \text{coin}_w(K) < \begin{cases} 
2^d \sqrt{e(d+1)}d(\ln d + \ln \ln d + 5) = O(2^d d^2 \ln d), & \text{if } K \text{ is } o\text{-symmetric}, \\
 e(d+1) \left( \left( \frac{2d}{d} \right)^{1/d} - 1 + \frac{d}{d+1} \right)^d d(\ln d + \ln \ln d + 5) \\
 = O(4^d d^{3/2} \ln d), & \text{otherwise}.
\end{cases} \]

Finally, in order to determine an upper bound on \( \text{coin} \), one only needs to apply (3.19) with \( \lambda = 1/2 \).
Corollary 3.7.3. Given $K \in \mathcal{K}^d$, $d \geq 2$, we have

$$
\text{coin}(K) \leq 2N_{1/2}(K) \leq \begin{cases} 
3^d(2d)(\ln d + \ln \ln d + 5) = O(3^d \ln d), & \text{if } K \text{ is } o\text{-symmetric}, \\
2^{d+1}\left(\left(\frac{2d}{d}\right)^{\frac{3}{2}} - \frac{1}{2}\right)^d d(\ln d + \ln \ln d + 5) = O(\tau^d \sqrt{d \ln d}), & \text{otherwise.}
\end{cases}
$$
Chapter 4

Contact number problem – the state of the art

In discrete geometry, the contact number of a finite packing of translates of a convex body (see Chapter 1) was introduced as a generalization of Newton’s kissing number. This notion has not only led to interesting mathematics, but has also found applications in the science of self-assembling materials, such as colloidal matter. With geometers, chemists, physicists and materials scientists researching the topic, there is a growing body of recent literature on this problem. Here, our aim is to give a snapshot of some of these developments. In Section 4.1, we briefly discuss the importance of the contact number problem in materials science. The next two sections are devoted to the known bounds on the contact number for $d = 2, 3$. Section 4.4, explores three computer-assisted empirical approaches that have been developed by applied scientists to estimate the contact numbers of packings of small numbers of unit spheres in $\mathbb{E}^3$. We analyze these approaches at length and show that despite being of interest, they fall short of providing exact values of largest contact numbers. Section 4.5 covers recent general results on packings of congruent balls and translates of an arbitrary convex body in $d$-space. Finally, the last section deals with the state of the contact number problem for non-congruent sphere packings.
4.1 Motivation from materials science

In addition to finding its origins in the works of pioneers like Newton, Erdős, Ulam and Fejes Tóth (see Section 4.2 for more on the role of latter two), the contact number problem is also important from an applications point of view. Packings of hard sticky spheres - impenetrable spheres with short-range attractive forces - provide excellent models for the formation of several real-life materials such as colloids, powders, gels and glasses [90]. The particles in these materials can be thought of as hard spheres that self-assemble into small and large clusters due to their attractive forces. This process, called self-assembly, is of tremendous interest to materials scientists, chemists, statistical physicists and biologists alike.

Of particular interest are colloids, which consist of particles at micron scale, dispersed in a fluid and kept suspended by thermal interactions [125]. Colloidal matter occurs abundantly around us - for example in glue, milk and paint. Moreover, controlled colloid formation is a fundamental tool used in scientific research to understand the phenomena of self-assembly and phase transition.

From thermodynamical considerations, it is clear that colloidal particles assemble so as to minimize the potential energy of the cluster. Since the range of attraction between these particles is extremely small compared to their sizes, two colloidal particles do not exert any force on each other until they are infinitessimally close, at which point there is strong attraction between them. As a result, they stick together, are resistant to drift apart, but strongly resistant to move any closer [5, 90]. Thus, two colloidal particles experiencing an attractive force from one another in a cluster can literally be thought of as being in contact.

It can be shown that under the force law described above, the potential energy of a colloidal cluster at reasonably low temperatures is inversely proportional to the number of contacts between its particles [5, 96, 98]. Thus, the particles are highly likely to assemble in packings that maximize the contact number. This has generated significant interest among materials scientists towards the contact number problem [5, 98], and has led to efforts in developing computer-assisted approaches to attack the problem. More details appear in
4.2 Largest contact numbers in the plane

4.2.1 The Euclidean plane

Harborth [89] proved the following well-known result on the contact graphs of congruent circular disk packings in \( \mathbb{E}^2 \).

**Theorem 4.2.1.** \( c(n, 2) = \left\lfloor 3n - \sqrt{12n - 3} \right\rfloor \), for all \( n \geq 2 \).

This result shows that an optimal way to pack \( n \) congruent disks to maximize their contacts is to pack them in a ‘hexagonal arrangement’. The arrangement starts by packing 6 unit disks around a central disk in such a way that the centers of the surrounding disks form a regular hexagon. The pattern is then continued by packing hexagonal layers of disks around the first hexagon. Thus the hexagonal packing arrangement, which is known to be the densest congruent disk packing arrangement, also achieves the maximum contact number \( c(n, 2) \), for all \( n \).

Interestingly, this also means that \( c(n, 2) \) equals the maximum number of sides that can be shared between \( n \) cells of a regular hexagon tiling of the plane. This connection was explored in [88], where isoperimetric hexagonal lattice animals of a given area \( n \) were explored. The connection between contact numbers and isoperimetric lattice animals is studied in detail in Chapter 5, so we skip the details here.

Despite the existence of a simple formula for \( c(n, 2) \), recognizing contact graphs of congruent disk packings is a challenging problem. The difficulty of this problem is made apparent by the following complexity result from [57].

**Theorem 4.2.2.** The problem of recognizing contact graphs of unit disk packings is NP-hard.

Quite surprisingly, the following rather natural stability version of Theorem 4.2.1 is still an open problem. (See also the final remarks in [55].)
Conjecture 4.2.3. There exists an $\epsilon > 0$ such that for any packing of $n$ circular disks of radii chosen from the interval $[1-\epsilon, 1]$ the number of touching pairs in the packing is at most $\lfloor 3n - \sqrt{12n - 3} \rfloor$, for all $n \geq 2$.

In 1984, Ulam (see [69]) proposed to investigate Erdős-type distance problems in normed spaces. Pursuing this idea, Brass [55] proved the following extension of Theorem 4.2.1 to normed planes.

Theorem 4.2.4. Let $K$ be a convex domain different from a parallelogram in $\mathbb{E}^2$. Then for all $n \geq 2$, one has $c(K, n, 2) = \lfloor 3n - \sqrt{12n - 3} \rfloor$. If $K$ is a parallelogram, then $c(K, n, 2) = \lfloor 4n - \sqrt{28n - 12} \rfloor$ for all $n \geq 2$.

The same idea inspired Bezdek [20] to investigate this question in $d$-space, details of which appear in Section 4.5.

Returning to normed planes, the following is a natural question.

Problem 4.2.5. Find an analogue of Theorem 4.2.4 for totally separable translative packings of convex domains in $\mathbb{E}^2$.

4.2.2 Spherical and hyperbolic planes

An analogue of Harborth’s theorem in the hyperbolic plane $\mathbb{H}^2$ was found by Bowen in [53]. In fact, his method extends to the 2-dimensional spherical plane $\mathbb{S}^2$. We prefer to quote these results as follows.

Theorem 4.2.6. Consider disk packings in $\mathbb{H}^2$ (resp., $\mathbb{S}^2$) by finitely many congruent disks, which maximize the number of touching pairs for the given number of congruent disks and of given diameter $D$. Then such a packing must have all of its centers located on the vertices of a triangulation of $\mathbb{H}^2$ (resp., $\mathbb{S}^2$) by congruent equilateral triangles of side length $D$ provided that the equilateral triangle in $\mathbb{H}^2$ (resp., $\mathbb{S}^2$) of side length $D$ has each of its angles equal to $\frac{2\pi}{N}$ for some positive integer $N \geq 3$. 

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In 1984, L. Fejes Tóth ([27]) raised the following attractive and related problem in $S^2$.
Consider an arbitrary packing $\mathcal{P}_r$ of disks of radius $r > 0$ in $S^2$. Let $\deg_{\text{avr}}(\mathcal{P}_r)$ denote the average degree of the vertices of the contact graph of $\mathcal{P}_r$. Then prove or disprove that $\limsup_{r \to 0} \left( \sup_{\mathcal{P}_r} \deg_{\text{avr}}(\mathcal{P}_r) \right) < 5$. This problem was settled in [27].

**Theorem 4.2.7.** Let $\mathcal{P}_r$ be an arbitrary packing of disks of radius $r > 0$ in $S^2$. Then

\[
\limsup_{r \to 0} \left( \sup_{\mathcal{P}_r} \deg_{\text{avr}}(\mathcal{P}_r) \right) < 5.
\]

We conclude this section with the still open hyperbolic analogue of Theorem 4.2.7 which was raised in [27].

**Conjecture 4.2.8.** Let $\mathcal{P}_r$ be an arbitrary packing $\mathcal{P}_r$ of disks of radius $r > 0$ in $\mathbb{H}^2$. Then

\[
\limsup_{r \to 0} \left( \sup_{\mathcal{P}_r} \deg_{\text{avr}}(\mathcal{P}_r) \right) < 5.
\]

### 4.3 Largest contact numbers in 3-space

Theorem 4.2.1 implies in a straightforward way that

\[
\lim_{n \to +\infty} \frac{3n - c(n, 2)}{\sqrt{n}} = \sqrt{12} = 3.464 \ldots .
\]  

(4.1)

Although one cannot hope for an explicit formula for $c(n, 3)$ in terms of $n$, there might be a way to prove a proper analogue of (4.1) in $\mathbb{E}^3$.

To this end we know only what is stated in Theorem 4.3.1. In order to state these results, we need an additional concept. Let us imagine that we generate packings of $n$ unit balls in $\mathbb{E}^3$ in such a special way that each and every center of the $n$ unit balls chosen is a lattice point of the face-centered cubic lattice with shortest non-zero lattice vector of length 2. Then let $c_{\text{fcc}}(n)$ denote the largest possible contact number of all packings of $n$ unit balls obtained in this way.
Figure 4.1: Contact graphs with \( c(n,3) \) contacts, for \( n = 1, 2, 3, 4, 5 \) (trivial cases) and largest known number of contacts, for \( n = 6, 7, 8, 9 \). For \( n = 1, 2, 3, 4, 5 \) the maximal contact graphs are unique up to isometry. All the packings listed are minimally rigid, and only for \( n = 9 \), the packing is not rigid as the two bipyramids can be twisted slightly about the common pivot (see Section 4.4).

The motivation for considering \( c_{\text{fcc}}(n) \) is obvious. Since in the planar case, the densest disk packing arrangement also maximizes contacts between disks and the face-centered cubic lattice is one of the densest for sphere packings in \( \mathbb{E}^3 \) \cite{[87]}, it makes sense to consider \( c_{\text{fcc}}(n) \) as a candidate for \( c(n,3) \). Moreover, it is easy to see that \( c_{\text{fcc}}(2) = c(2,3) = 1, c_{\text{fcc}}(3) = c(3,3) = 3 \) and \( c_{\text{fcc}}(4) = c(4,3) = 6 \).

**Theorem 4.3.1.**

(i) \( c(n,3) < 6n - 0.926n^{\frac{3}{2}} \), for all \( n \geq 2 \).

(ii) \( c_{\text{fcc}}(n) < 6n - \frac{3\sqrt{18\pi}}{\pi} n^{\frac{3}{2}} = 6n - 3.665 \ldots n^{\frac{3}{2}} \), for all \( n \geq 2 \).

(iii) \( 6n - \sqrt{18} n^{\frac{3}{2}} < 2k(2k^2 - 3k + 1) \leq c_{\text{fcc}}(n) \leq c(n,3) \), for all \( n = \frac{k(2k^2+1)}{3} \) with \( k \geq 2 \).

Recall that (i) was proved in \cite{[35]} (using the method of \cite{[23]}), while (ii) and (iii) were
proved in [23]. Clearly, Theorem 4.3.1 implies that

\[ 0.926 < \frac{6n - c(n, 3)}{n^{\frac{3}{2}}} < \sqrt[3]{486} = 7.862 \ldots, \] (4.2)

for all \( n = \frac{k(2k^2+1)}{3} \) with \( k \geq 2 \).

Now consider the complexity of recognizing contact graphs of congruent sphere packings in \( \mathbb{E}^3 \). Just like its 2-dimensional analogue, Hliněný [94] showed the 3-dimensional problem to be NP-hard by reduction from 3-SAT. In fact, the same is true in four dimensions [94].

**Theorem 4.3.2.** The problem of recognizing contact graphs of unit sphere packings in \( \mathbb{E}^3 \) and \( \mathbb{E}^4 \) is NP-hard.

### 4.4 Empirical approaches

Throughout this section, we deal with finite unit sphere packings in three dimensional Euclidean space, that is, with finite packings of unit balls in \( \mathbb{E}^3 \). Therefore, in this section a ‘sphere’ always means a unit sphere in \( \mathbb{E}^3 \). Taking inspiration from materials science and statistical physics, we will often refer to a finite sphere packing as a *cluster*. Our aim is to describe three computational approaches that have recently been employed in constructing putatively maximal contact graphs for packings of \( n \) spheres under certain rigidity assumptions.

**Definition 4.4.1** (Minimal rigidity [5]). A cluster of \( n \geq 4 \) unit spheres is said to be *minimally rigid* if

- each sphere is in contact with at least 3 others, and
- the cluster has at least \( 3n - 6 \) contacts (that is, the corresponding contact graph has at least \( 3n - 6 \) edges).

**Definition 4.4.2** (Rigidity [96]). A cluster of \( n \) unit spheres is (nonlinearly) rigid if it cannot be deformed continuously by any finite amount and still maintain all contacts.
The first two approaches - which we discuss together - deal with minimally rigid clusters, while the third investigates rigid clusters. We observe that one can find minimally rigid clusters that are not rigid. One such example for \( n = 9 \) is given in [5, Fig. 1 (i)].

4.4.1 Contact number estimates for up to 11 spheres

Arkus, Manoharan and Brenner [5] made an attempt to exhaustively generate all minimally rigid packings of \( n \) spheres that are either local or global maxima of the number of contacts. Here, a packing is considered a global maximum if the spheres in the cluster cannot form any additional contacts or a local maximum if new contacts can only be created after breaking an existing contact. They produce a list of maximal contact minimally rigid sphere packings for \( n = 2, \ldots, 9 \), which is putatively complete up to possible omissions due to round off errors, and a partial list for 10 spheres. Since the number of such packings grows exponentially with \( n \), their approach can only be implemented on a computer.

Before we delve into the details of their methodology, it would be pertinent to understand why it focuses on finding minimally rigid clusters. It seems the minimal rigidity was considered due to two reasons, the first being Maxwell’s criterion [131], which is popular in physics literature and states that a rigid cluster of \( n \) spheres has at least \( 3n - 6 \) contacts. This is false since examples of rigid clusters with \( n \geq 10 \) have been reported [96] that are not minimally rigid. The second reason is the intuition that any maximum contact cluster of \( n \geq 4 \) spheres should be minimally rigid. To our knowledge, there exists no proof of or counterexample to this intuition. We can, however, prove the following. The proof depends on the assumption that Arkus et al. [5] have found all minimally rigid packings of \( n \leq 9 \) spheres that maximize the number of contacts.

**Proposition 4.4.3.** Assume that all maximal contact minimally rigid packings of \( n \leq 9 \) spheres are listed in [4] and [5]. Then for \( n = 4, \ldots, 9 \),

\[
c(n, 3) = 3n - 6,
\]
and there exists a minimally rigid cluster with $c(n, 3)$ contacts.

Proof. We introduce some terminology for finite sphere packings and their contact graphs. We say that any three pairwise touching spheres form a triangle. A triangle is called an exposed triangle if an additional sphere, not part of the original packing, can be brought in contact with all three spheres in the triangle without overlapping with any sphere already in the packing. Triangles and exposed triangles can be equivalently defined for contact graphs.

Claim: Any maximal contact graph on $n$ vertices with $4 \leq n \leq 9$ has an exposed triangle and each vertex of such a graph has degree at least 3.

By checking the list of all minimally rigid packings of $4 \leq n \leq 9$ spheres given in [4] exhaustively, we see that the claim holds for all such sphere packings. We now proceed by induction on $n$.

For $n = 4$, there is only one maximal contact graph, and for that graph the Claim holds. Now suppose the Claim holds for some $n \geq 4$. Consider any contact graph $G$ that has the largest number of contacts among all contact graphs having $n + 1$ vertices. Let $v$ be a vertex of $G$.

Suppose that $v$ has degree 2. Then $G - v$, the graph obtained by deleting $v$ and all edges incident to $v$ from $G$, must be a maximal contact graph on $n$ vertices, since if this is not the case, then replacing $G - v$ by a maximal contact graph $H$ on $n$ vertices and joining $v$ to any exposed triangle of $H$ produces a contact graph on $n + 1$ vertices with strictly more contacts than $G$. But then $G - v$ has an exposed triangle and joining $v$ to that triangle produces a contact graph on $n + 1$ vertices with strictly more contacts than $G$. This is a contradiction and so $v$ has degree at least 3. Thus $G$ is minimally rigid and has an exposed triangle. This completes the proof of Claim.

Thus, for $n = 4, \ldots, 9$, the list of all maximal contact graphs coincides with the list of minimally rigid maximal contact graphs that have $3n - 6$ contacts according to [5].

---

1 The complete list (up to possible omissions due to round off errors) of minimally rigid packings of $n \leq 9$ spheres and a preliminary list of $n = 10$ spheres appears on the arXiv [4]. The paper [5] only contains a partial list, so for the more complete list we refer to the arXiv version.
We now describe the approach of Arkus et al. [5]. Note that since we are dealing with unit spheres (as stated in the opening of this section), the distance between the centers of two touching spheres is 2. Let $n \geq 4$ be a positive integer.

**Procedure 1 ([5]):**

**Step 1:** List the adjacency matrices of all nonisomorphic simple graphs with $n$ vertices and exactly $3n - 6$ edges such that each vertex has degree at least 3. Let $\mathcal{A}$ be the set of all such adjacency matrices. In [5], this step is performed using the graph isomorphism testing program *nauty* and the Sage package *nice*.

**Step 2:** For each $A \in \mathcal{A}$, there is a corresponding simple graph $G_A$ with vertex set (say) $V = \{v_1, \ldots, v_n\}$. Denote the $(i,j)$-entry of $A$ by $A_{ij}$ and consider each vertex $v_i$ of $G_A$ as a point $v_i = (x_i, y_i, z_i)$ (the coordinates are yet unknown) in $\mathbb{E}^3$. Then $G_A$ is a contact graph if and only if we can place congruent spheres centered at the vertices of $G_A$ such that none of the spheres overlap and $A_{ij} = 1$ implies that the spheres centered at $v_i$ and $v_j$ touch. Use the simple geometric elimination rules derived in [5] to remove a substantial number of adjacency matrices from $\mathcal{A}$ that cannot be realized into contact graphs. These geometric rules basically detect certain patterns that cannot occur in the adjacency matrices of contact graphs. Let the resulting set of adjacency matrices be denoted by $\mathcal{B}$.

**Step 3:** For any $A \in \mathcal{B}$, $G_A$ is a contact graph of a unit sphere packing if and only if for $i > j$ the system of nonlinear equations

$$
D_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = 2^2 = 4, \quad A_{ij} = 1,
$$

$$
D_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \geq 2^2 = 4, \quad A_{ij} = 0.
$$

has a real solution. Note that we are only considering $i > j$ as the matrix $A$ is symmetric. Without loss of generality, we can assume that $x_1 = y_1 = z_1 = 0$ (the first sphere is centered at the origin); $y_2 = z_2 = 0$ (the second sphere lies on the $x$-axis) and $z_3 = 0$ (the third sphere lies in the $xy$-plane). Therefore, we obtain a system with $\frac{n(n-1)}{2}$ constraints (of which $3n - 6$
are equality constraints) in $3n - 6$ unknowns. Here $D_{ij}$ is the distance between vertices $v_i$ and $v_j$. In [5], for each $A \in \mathcal{B}$, the system (4.3) is solved analytically for $n \leq 9$ and numerically for $n = 10$.

**Step 4**: Form the distance matrix $D_A = [D_{ij}]$ and let $\mathcal{D}$ be the set of all distance matrices corresponding to valid contact graphs of packings of $n$ unit spheres. The contact number corresponding to any $D \in \mathcal{D}$ equals the number of entries of $D$ that equal 2 and lie above (equivalently below) the main diagonal. Note that, although we started with the adjacency matrices corresponding to exactly $3n - 6$ contacts, solving system (4.3) yields all distance matrices with $3n - 6$ or more contacts.

Since the geometric rules used in Step 2 are susceptible to round off errors, there is a possibility that some adjacency matrices are incorrectly eliminated from $\mathcal{A}$. Also, for $n = 10$, Newton’s method is used to solve (4.3) as the computational limit of analytical methods was reached for packings of 10 spheres. Thus, the list of minimally rigid sphere packings provided in [5] could potentially be incomplete. As a result, the contact number $c(n, 3)$ is still unknown for $n \geq 6$.\footnote{According to [5], for $n \leq 7$, it is possible to solve the system (4.3) using standard algebraic geometry methods for all $A \in \mathcal{A}$ without filtering by geometric rules. Arkus et al. [5] attempted this using the package SINGULAR. Therefore, most likely for $n = 6, 7$, the maximal contact graphs as obtained in [5] are optimal for minimally rigid sphere packings.}

Nevertheless, it is quite reasonable to conjecture the following.

**Conjecture 4.4.4.** For $n \geq 6$, every contact graph of a packing of $n$ spheres with $c(n, 3)$ contacts is minimally rigid. Moreover, for $n = 6, \ldots, 9$,

$$c(n, 3) = 3n - 6.$$  

Hoy et al. [98] extended Procedure 1 to 11 spheres. However, they employ Newton’s method, which is not guaranteed to obtain a solution, even when one exists. Also they make the erroneous assumption that the contact graph of any minimally rigid sphere packing contains a Hamiltonian path. This assumption greatly reduces the number of adjacency
matrices to be considered. However, Connelly, E. Demaine and M. Demaine show this to be false [91], providing a counterexample with 16 vertices.

4.4.2 Maximal contact rigid clusters

Despite its intuitive significance, we have seen that minimal rigidity is neither sufficient nor necessary for rigidity. Holmes-Cerfon [96] developed a computational technique to potentially construct all rigid sphere packings of a small number of spheres. Her idea to consider rigid clusters comes from the intuition that in a physical system (like self-assembling colloids), a rigid cluster is more likely to form and survive than a non-rigid cluster.

Some aspects of Holmes-Cerfon’s method are similar to the empirical approaches described earlier. For instance, the mathematical formulation in terms of adjacency and distance matrices, and use of system (4.3) to arrive at potential solutions remains unchanged. However, there are two fundamental differences.

Obviously, one is the consideration of rigidity instead of minimal rigidity. This results in the removal of the restriction that the contact graph should have at least $3n - 6$ edges. Instead, any solution obtained is tested for rigidity.

The second major difference lies in the way all potential solutions are reached. In the previous approaches, the method involved an exhaustive adjacency matrix search followed by filtering through some geometrical rules. Here, the procedure starts with a single rigid packing $\mathcal{P}$ of $n$ spheres and attempts to generate all other rigid packings of $n$ spheres as follows. First, break an existing contact in $\mathcal{P}$ by deleting an equation from (4.3). This usually leads to a single internal degree of freedom that results in a one-dimensional solution set. When this happens, one can follow the one-dimensional path numerically until another contact is formed, typically resulting in another rigid packing [96].

The paper [96] provides a preliminary list for all rigid sphere packings of up to 14 spheres, and maximal contact packings of up to 19 spheres. However, the use of numerical methods and approximations throughout means that the list is potentially incomplete.
Table 4.1 lists the known values, bounds and empirical estimates of maximal contact number of sphere packings in $\mathbb{E}^3$. The second column lists the lower bound when $n$ equals an octahedral number, i.e., $n = \frac{k(2k^2+1)}{3}$, for some $k = 2, 3, \ldots$ [23]. The third column lists the upper bound for packings on the face-centered cubic (fcc) lattice for all $n$ [23], while the fourth column contains the general upper bound for all $n$ [35]. The last column contains the trivially known exact values for $n = 2, 3, 4, 5$ and the largest contact numbers found by the empirical approaches (for $n = 6, 7, 8, 9, 10$ from [5], for $n = 11$ from [98] and for $n = 12, \ldots, 19$ from [96]). An asterisk '*' in the last column indicates the largest known contact number for minimally rigid clusters, while a double asterisk '***' indicates the largest known contact number for rigid clusters.

Table 4.1: Bounds on the contact numbers of sphere packings in 3-space.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Lower bound $2k(2k^2 - 3k + 1)$</th>
<th>fcc upper bound $\left\lfloor 6n - \frac{3\sqrt{18\pi}}{\pi}n^{2/3} \right\rfloor$</th>
<th>General upper bound $\left\lfloor 6n - 0.926n^{2/3} \right\rfloor$</th>
<th>Exact values/ Putatively largest</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>10</td>
<td></td>
<td>1 ($= 3n - 5$)(trivial)</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>16</td>
<td></td>
<td>3 ($= 3n - 6$)(trivial)</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>21</td>
<td></td>
<td>6 ($= 3n - 6$)(trivial)</td>
</tr>
<tr>
<td>5</td>
<td>19</td>
<td>27</td>
<td></td>
<td>9 ($= 3n - 6$)(trivial)</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>23</td>
<td>32</td>
<td>12* ($= 3n - 6$)</td>
</tr>
<tr>
<td>7</td>
<td>28</td>
<td>38</td>
<td></td>
<td>15* ($= 3n - 6$)</td>
</tr>
<tr>
<td>8</td>
<td>33</td>
<td>44</td>
<td></td>
<td>18* ($= 3n - 6$)</td>
</tr>
<tr>
<td>9</td>
<td>38</td>
<td>49</td>
<td></td>
<td>21* ($= 3n - 6$)</td>
</tr>
<tr>
<td>10</td>
<td>42</td>
<td>55</td>
<td></td>
<td>25* ($= 3n - 5$)</td>
</tr>
<tr>
<td>11</td>
<td>47</td>
<td>61</td>
<td></td>
<td>29* ($= 3n - 4$)</td>
</tr>
<tr>
<td>12</td>
<td>52</td>
<td>67</td>
<td></td>
<td>33** ($= 3n - 3$)</td>
</tr>
<tr>
<td>13</td>
<td>57</td>
<td>72</td>
<td></td>
<td>36** ($= 3n - 3$)</td>
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<td>14</td>
<td>62</td>
<td>78</td>
<td></td>
<td>40** ($= 3n - 2$)</td>
</tr>
<tr>
<td>15</td>
<td>67</td>
<td>84</td>
<td></td>
<td>44** ($= 3n - 1$)</td>
</tr>
<tr>
<td>16</td>
<td>72</td>
<td>90</td>
<td></td>
<td>48** ($= 3n$)</td>
</tr>
<tr>
<td>17</td>
<td>77</td>
<td>95</td>
<td></td>
<td>52** ($= 3n + 1$)</td>
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<td>101</td>
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<td>56** ($= 3n + 2$)</td>
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<td>19</td>
<td>60</td>
<td>87</td>
<td>107</td>
<td>60** ($= 3n + 3$)</td>
</tr>
</tbody>
</table>
4.5 On largest contact numbers in higher dimensional spaces

In this section, we study the contact number problem in \( \mathbb{E}^d \), both for packings of \( B^d \) and translates of an arbitrary \( d \)-dimensional convex body \( K \).

4.5.1 Packings by translates of a convex body

One of the main results of this section is an upper bound for the number of touching pairs in an arbitrary finite packing of translates of a convex body, proved in [20]. In order to state the theorem in question in a concise way we need a bit of notation. Let \( K \) be an arbitrary convex body in \( \mathbb{E}^d, \ d \geq 3 \), then let \( \delta(K) \) denote the density of a densest packing of translates of the convex body \( K \) in \( \mathbb{E}^d, \ d \geq 3 \). Moreover, let

\[
iq(K) := \frac{(s\text{vol}_{d-1}(bdK))^{\frac{d}{d-1}}}{(\text{vol}_d(K))^{\frac{d}{d-1}}}\]

be the isoperimetric quotient of the convex body \( K \), where \( s\text{vol}_{d-1}(bdK) \) denotes the \((d-1)\)-dimensional surface volume of the boundary \( bdK \) of \( K \), and \( \text{vol}_d(K) \) denotes the \( d \)-dimensional volume of \( K \). Furthermore, let \( H(K) \) denote the Hadwiger number of \( K \), which is the largest number of non-overlapping translates of \( K \) that can all touch \( K \). An elegant observation of Hadwiger [84] is that \( H(K) \leq 3^d - 1 \), where equality holds if and only if \( K \) is an affine \( d \)-cube. Finally, let the one-sided Hadwiger number \( h(K) \) of \( K \) be the largest number of non-overlapping translates of \( K \) that touch \( K \) and that all lie in a closed supporting halfspace of \( K \). In [26], using the Brunn–Minkowski inequality, it is proved that \( h(K) \leq 2 \cdot 3^{d-1} - 1 \), where equality is attained if and only if \( K \) is an affine \( d \)-cube. Let \( K_o := \frac{1}{2}(K + (-K)) \) be the normalized (centrally symmetric) difference body assigned to \( K \).
Theorem 4.5.1 ([20]). Let $K$ be an arbitrary convex body in $\mathbb{E}^d$, $d \geq 3$. Then

$$c(K, n, d) \leq \frac{H(K_o)}{2} n - \frac{1}{2^d \delta(K_o) \pi^\frac{d-1}{2}} \sqrt{\frac{\omega_d}{\omega_d}} n^{\frac{d-1}{d}} - (H(K_o) - h(K_o) - 1)$$

$$\leq \frac{3^d - 1}{2} n - \frac{\sqrt{\omega_d}}{2^{d+1}} n^{\frac{d-1}{d}},$$

where $\omega_d = \frac{\pi^\frac{d}{2}}{\Gamma(\frac{d}{2} + 1)} = \text{vol}_d(B^d)$.

Since for the most part we are interested in contact numbers of sphere packings, it would be interesting to see the form Theorem 4.5.1 takes when $K = B^d$. Recall that $k(d)$ denotes the kissing number of a unit ball in $\mathbb{E}^d$. Let $\delta_d$ stand for the largest possible density for (infinite) packings of unit balls in $\mathbb{E}^d$. The following consequence of Theorem 4.5.1 was reported in [23].

Corollary 4.5.2. Let $n > 1$ and $d \geq 3$ be positive integers. Then

$$c(n, d) < \frac{1}{2} k(d) n - \frac{1}{2^d \delta_d} \frac{d-1}{d} n^{\frac{d-1}{d}}.$$

Now, recall the well-known theorem of Kabatiansky and Levenshtein [102] that $k(d) \leq 2^{0.401d(1+o(1))}$ and $\delta_d \leq 2^{-0.599d(1+o(1))}$ as $d \to +\infty$. Together with Corollary 4.5.2 this gives

$$c(n, d) < \frac{1}{2} 2^{0.401d(1+o(1))} n - \frac{1}{2^d} 2^{0.599(1+o(1))(d-1)} n^{\frac{d-1}{d}},$$

for $n > 1$, as $d \to +\infty$.

In particular, for $d = 3$ we have $k(3) = 12$ [155] and $\delta_3 = \frac{\pi}{\sqrt{18}}$ [87]. Thus, by combining these with Corollary 4.5.2 we find that for $n > 1$,

$$c(n, 3) < 6n - \frac{1}{8} \left( \frac{\pi}{\sqrt{18}} \right)^{-\frac{2}{3}} n^\frac{2}{3} = 6n - 0.152 \ldots n^\frac{2}{3}.$$

The above upper bound for $c(n, 3)$ was substantially improved, first in [23] and then further in [35]. The current best upper bound is stated in Theorem 4.3.1 (i).
In the proof of Theorem 4.5.1 published in [20], the following statement plays an important role that might be of independent interest, and so we quote it as follows. For the sake of completeness, we wish to point out that Theorem 4.5.3 and Corollary 4.5.4 are actual strengthenings of Theorem 3.1 and Corollary 3.1 of [13] mainly because, in our case, the containers of the packings in question are highly non-convex.

**Theorem 4.5.3.** Let $K_o$ be a convex body in $\mathbb{E}^d$, $d \geq 2$, symmetric about the origin $o$ of $\mathbb{E}^d$ and let $\{c_1 + K_o, c_2 + K_o, \ldots, c_n + K_o\}$ be an arbitrary packing of $n > 1$ translates of $K_o$ in $\mathbb{E}^d$. Then

$$\frac{n \text{vol}_d(K_o)}{\text{vol}_d(\bigcup_{i=1}^{n}(c_i + 2K_o))} \leq \delta(K_o).$$

The following is an immediate corollary of Theorem 4.5.3.

**Corollary 4.5.4.** Let $\mathcal{P}_n(K_o)$ be the family of all possible packings of $n > 1$ translates of the $o$-symmetric convex body $K_o$ in $\mathbb{E}^d$, $d \geq 2$. Moreover, let

$$\delta(K_o, n) := \max \left\{ \frac{n \text{vol}_d(K_o)}{\text{vol}_d(\bigcup_{i=1}^{n}(c_i + 2K_o))} \mid \{c_1 + K_o, \ldots, c_n + K_o\} \in \mathcal{P}_n(K_o) \right\}.$$

Then

$$\limsup_{n \to \infty} \delta(K_o, n) = \delta(K_o).$$

Interestingly enough, one can interpret the contact number problem on the exact values of $c(n,d)$ as a volume minimization question. Here, we give only an outline of that idea introduced and discussed in detail in [32].

**Definition 4.5.5.** Let $\mathcal{P}^n := \{c_i + B^d \mid 1 \leq i \leq n \text{ with } \|c_j - c_k\| \geq 2 \text{ for all } 1 \leq j < k \leq n\}$ be an arbitrary packing of $n > 1$ unit balls in $\mathbb{E}^d$. The part of space covered by the unit balls of $\mathcal{P}^n$ is labelled by $\mathcal{P}^n := \bigcup_{i=1}^{n}(c_i + B^d)$. Moreover, let $C^n := \{c_i \mid 1 \leq i \leq n\}$ denote the set of centers of the unit balls in $\mathcal{P}^n$. Furthermore, for any $\lambda > 0$ let $\mathcal{P}^n_\lambda := \bigcup \{x + \lambda B^d \mid x \in \mathcal{P}^n\} = \bigcup_{i=1}^{n}(c_i + (1 + \lambda)B^d)$ denote the outer parallel domain of $P^n$ having outer radius $\lambda$. 81
Finally, let
\[
\delta_d(n, \lambda) := \max_{\mathcal{P}^n} \frac{n \omega_d}{\operatorname{vol}_d(P^n)} = \frac{n \omega_d}{\min_{\mathcal{P}^n} \operatorname{vol}_d(\bigcup_{i=1}^n (c_i + (1 + \lambda) B^d))}
\]
\[
\delta_d(\lambda) := \limsup_{n \to +\infty} \delta_d(n, \lambda).
\]

Now, let \( \mathcal{P} := \{c_i + B^d \mid i = 1, 2, \ldots \text{ with } \|c_j - c_k\| \geq 2 \text{ for all } 1 \leq j < k \} \) be an arbitrary infinite packing of unit balls in \( \mathbb{E}^d \). Recall that the packing density \( \delta_d \) of unit balls in \( \mathbb{E}^d \) can be computed as follows:
\[
\delta_d = \sup_{\mathcal{P}} \left( \limsup_{R \to +\infty} \frac{\sum_{c_i + B^d \subset R B^d} \operatorname{vol}_d(c_i + B^d)}{\operatorname{vol}_d(R B^d)} \right).
\]

Hence, it is rather easy to see that \( \delta_d \leq \delta_d(\lambda) \) holds for all \( \lambda > 0, d \geq 2 \). On the other hand, it was proved in [20] (see also Corollary 4.5.4) that \( \delta_d = \delta_d(\lambda) \) for all \( \lambda \geq 1 \) leading to the classical sphere packing problem. Now, we are ready to put forward the following question from [32].

**Problem 4.5.6.** Determine (resp., estimate) \( \delta_d(\lambda) \) for \( d \geq 2, 0 < \lambda < \sqrt{\frac{2d}{d+1}} - 1 \).

First, we note that \( \frac{2}{\sqrt{3}} - 1 \leq \sqrt{\frac{2d}{d+1}} - 1 \) for all \( d \geq 2 \). Second, observe that as \( \frac{2}{\sqrt{3}} \) is the circumradius of a regular triangle of side length 2, so if \( 0 < \lambda < \frac{2}{\sqrt{3}} - 1 \), then for any unit ball packing \( \mathcal{P}^n \) no three of the closed balls in the family \( \{c_i + (1 + \lambda) B^d \mid 1 \leq i \leq n\} \) have a point in common. In other words, for any \( \lambda \) with \( 0 < \lambda < \frac{2}{\sqrt{3}} - 1 \) and for any unit ball packing \( \mathcal{P}^n \), in the arrangement \( \{c_i + (1 + \lambda) B^d \mid 1 \leq i \leq n\} \) of closed balls of radii \( 1 + \lambda \), only pairs of balls may overlap. Thus computing \( \delta_d(n, \lambda) \), i.e., minimizing \( \operatorname{vol}_d(P^n) \), means maximizing the total volume of pairwise overlaps in the ball arrangement \( \{c_i + (1 + \lambda) B^d \mid 1 \leq i \leq n\} \) with the underlying packing \( \mathcal{P}^n \). Intuition would suggest achieving this by simply maximizing the number of touching pairs in the unit ball packing \( \mathcal{P}^n \). Hence, Problem 4.5.6 becomes very close to the contact number problem of finite unit ball packings for \( 0 < \lambda < \frac{2}{\sqrt{3}} - 1 \). Indeed, we have the following statement proved in [32].
Theorem 4.5.7. Let \( n > 1 \) and \( d > 1 \) be given. Then there exists \( \lambda_{d,n} > 0 \) and a packing \( \hat{P}^n \) of \( n \) unit balls in \( \mathbb{E}^d \) possessing the largest contact number for the given \( n \) such that for all \( \lambda \) satisfying \( 0 < \lambda < \lambda_{d,n} \), \( \delta_d(n, \lambda) \) is generated by \( \hat{P}^n \), i.e., \( \text{vol}_d(P^n_{\lambda}) \geq \text{vol}_d(\hat{P}^n_{\lambda}) \) holds for every packing \( P^n \) of \( n \) unit balls in \( \mathbb{E}^d \).

4.5.2 Contact graphs of unit sphere packings in \( \mathbb{E}^d \)

Given the NP-hardness of recognizing contact graphs of unit sphere packings for \( d = 2, 3, 4 \), Hliněný [94] conjectured that the problem remains NP-hard in any fixed dimension.

Conjecture 4.5.8. The recognition of contact graphs of unit sphere packings is NP-hard in any fixed dimension \( d \geq 2 \).

Hliněný and Kratochvíl [95] made some progress towards this conjecture. They reproved Theorem 4.3.2 using the rather elaborate notion of a scheme of an \( m \)-comb and then proved Conjecture 4.5.8 for \( d = 3, 4, 8, 24 \). To define an \( m \)-comb we need to introduce some more terminology.

For a hyperplane \( h \) in \( \mathbb{E}^d \) and \( S \subseteq \mathbb{E}^d \), let \( S/h \) denote the mirror reflection of \( S \) across \( h \). We say that a set \( S \) is a minimal-distance representation of a graph \( G \), denoted by \( G = M(S) \), if the vertices of \( G \) are the points of \( S \), and the edges of \( G \) correspond to minimal-distance pairs of points in \( S \). The graph \( G \) is then called the minimal-distance graph of \( S \). Also, let \( m(S) \) denote the minimal distance among pairs of points of \( S \). Finally, when \( m(S) = 1 \), we say that the set \( S \) is rigid in \( \mathbb{E}^d \) if for any set \( S' \subseteq \mathbb{E}^d \) with \( m(S') = 1 \), the following holds: if \( \phi : M(S') \to M(S) \) is an isomorphism, then \( \phi \) is an isometry of the underlying sets \( S', S \). (Notice that the definition of a rigid set is slightly stronger than just saying that \( S \) has a unique representation up to isometry.) For example, the vertices of a regular tetrahedron or a regular octahedron form rigid sets in \( \mathbb{E}^3 \). In general, the vertices of a \( d \)-dimensional simplex or a \( d \)-dimensional cross-polytope are rigid sets in \( \mathbb{E}^d \).

Definition 4.5.9 (Scheme of an \( m \)-comb [95]). Let \( T, V, W \) be point sets in \( \mathbb{E}^d \), and let \( \alpha, \beta \)
be vectors in $\mathbb{E}^d$. The five-tuple $(V, W, T, \alpha, \beta)$ is called a scheme of an $m$-comb in $\mathbb{E}^d$ if the following conditions are satisfied:

- The sets $V \cup W$ and $T$ are both rigid in $\mathbb{E}^d$, and $m(V) = m(V \cup W) = m(T) = 1$.
- The set $V$ spans a hyperplane $h$ in $\mathbb{E}^d$. The vector $\alpha$ is parallel to $h$. Let $T_0 = T \cap (T - \beta)$. Then the set $T_0$ spans $\mathbb{E}^d$. For $i = 0, \ldots, m - 1$, the set $(T_0 + i\alpha) \cap V$ spans the hyperplane $h$.
- Let $c$ be the maximal distance of $W$ from $h$. Then the distance between $h$ and $h + \beta$ is greater than $2c + 1$. The distance between $h$ and $T + \beta$ or $T - 2\beta$ is greater than $c + 1$.
- Let $p$ be the straight line parallel to $\beta$ such that the maximal distance $c'$ between $p$ and the points of $T$ is minimized. Then the distance between $p$ and $p + \alpha$ is greater than $2c' + 1$. For $j \in \mathbb{Z} - \{0, \ldots, m - 1\}$, the distance between the sets $V \cup W$ and $p + j\alpha$ is greater than $c' + 1$.
- The sets $T$ and $(T - \beta) \setminus T$ are non-overlapping, and the sets $T$ and $T + 2\beta$ are strictly non-overlapping, while the sets $T$ and $(T/h) + \beta$ are overlapping. Let $T' = T \cup (T - \beta)$. Then for $i = 0, \ldots, m - 1$, the sets $V$ and $(T' + i\alpha) \setminus V$ are non-overlapping.

The term ‘$m$-comb’ comes from the actual geometry of such a scheme, which is comb-like (see the illustration of an $m$-comb in [95]). It turns out that if $d \geq 3$ is such that for every $m > 0$, there exists a scheme of an $m$-comb in $\mathbb{E}^d$, then the recognition of contact graphs of unit sphere packings in $\mathbb{E}^d$ is an NP-hard problem [95].

**Theorem 4.5.10.** The problem of recognizing contact graphs of unit sphere packings is NP-hard in $\mathbb{E}^3$, $\mathbb{E}^4$, $\mathbb{E}^8$ and $\mathbb{E}^{24}$.

The proof relies on constructing such schemes for $d = 3, 4, 8, 24$. For $d \neq 2, 3, 4, 8, 24$, the complexity of recognizing unit sphere contact graphs is unknown, while for $d = 2$ it is NP-hard from Theorem 4.2.2.
4.6 Contact graphs of non-congruent sphere packings

So far, we have exclusively focused on contact graphs of packings of congruent spheres. In this section, we discuss what is known for general non-congruent sphere packings. Let us denote by $c^*(n, d)$ the maximal number of edges in a contact graph of $n$ not necessarily congruent $d$-dimensional spheres. Clearly, $c^*(n, d) \geq c(n, d)$ for any positive integers $n$ and $d \geq 2$.

The planar case was first resolved by Koebe [111] in 1936. Koebe’s result was later rediscovered by Andreev [3] in 1970 and by Thurston in [164] 1978. The result is referred to as Koebe–Andreev–Thurston theorem or the circle packing theorem.

In terms of contact graphs, the result can be stated as below.

**Theorem 4.6.1** (Koebe–Andreev–Thurston). A graph $G$ is a contact graph of a (not necessarily congruent) circle packing in $\mathbb{E}^2$ if and only if $G$ is planar.

In other words, for any planar graph $G$ of any order $n$, there exist $n$ circular disks with possibly different radii such that when these disks are placed with their centers at the vertices of the graph, the disks centered at the end vertices of each edge of $G$ touch. In addition, this cannot be achieved for any nonplanar graph. This is a rather unique result that is, as we will see shortly, highly unlikely to have an analogue in higher dimensions. It shows that $c^*(n, 2) = 3n - 6$ for $n \geq 2$, which is the number of edges in a maximal planar graph.

A similar simple characterization of contact graphs of general not necessarily congruent sphere packings cannot be found for all dimensions $d \geq 3$, unless P = NP. We briefly discuss this here. In [94, 95], the authors report that Kirkpatrick and Rote informed them of the following result in a personal communication in 1997. The proof appears in [95].

**Theorem 4.6.2.** A graph $G$ has a $d$-unit-ball contact representation if and only if the graph $G \oplus K_2$ has a $(d + 1)$-ball contact representation.

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3It is worth-noting that Koebe’s paper was written in German and titled ‘Kontaktprobleme der konformen Abbildung’ (Contact problems of conformal mapping). Andreev’s paper appeared in Russian. Probably, the first instance of this result appearing in English was in Thurston’s lecture notes that were distributed by Princeton University in 1980. However, the lectures were delivered in 1978-79 [164].
Here $K_2$ denotes the complete graph on two vertices, while $G \oplus H$ represents the graph formed by taking the disjoint union of $G$ and $H$ and then adding all edges across [95]. Theorem 4.6.2 provides an interesting connection between contact graphs of unit sphere packings in $E^d$ and contact graphs of not necessarily congruent sphere packings in $E^{d+1}$. Combining this with Theorem 4.2.2, 4.3.2 and 4.5.10 gives the following [95].

**Corollary 4.6.3.** The problem of recognizing general contact graphs of (not necessarily congruent) sphere packings is NP-hard in dimensions $d = 3, 4, 5, 9, 25$.

Not much is known about $c^*(n, d)$, for $d \geq 4$. However, for $d = 3$, an upper bound was found by Kuperberg and Schramm [114]. Define the average kissing number $k^*_\text{av}(d)$ in dimension $d$ as the supremum of average vertex degrees among all contact graphs of finite sphere packings in $E^d$. In a packing of three dimensional congruent spheres, a sphere can touch at the most 12 others [155]. Thus, a three dimensional ball $B$ cannot touch more than 12 other balls at least as large as $B$. It follows that $k^*_\text{av}(3) \leq 2k(3) = 24$. In [114], this is improved to $12.566 \approx 666/53 \leq k^*_\text{av}(3) < 8 + 4\sqrt{3} \approx 14.928$. In the language of contact numbers, the Kuperberg–Schramm bound translates into the following.

**Theorem 4.6.4.** $c^*(n, 3) < (4 + 2\sqrt{3})n \approx 7.464n$.

The method of Kuperberg and Schramm relies heavily on the geometry of 3-dimensional space. As a result it seems difficult to generalize it to higher dimensions. We close this section with the following open question.

**Problem 4.6.5.** Find upper and lower bounds on $c^*(n, d)$ in the spirit of Kuperberg–Schramm bounds on $c^*(n, 3)$.
Chapter 5

Hadwiger and contact numbers of totally separable domains and their crystallization

Here, we investigate the Hadwiger and contact numbers of totally separable packings of convex domains, which we refer to as the separable Hadwiger number and the separable contact number, respectively. We show that the separable Hadwiger number of any smooth convex domain is 4 and the maximum separable contact number of any packing of $n$ translates of a smooth strictly convex domain is $\lfloor 2n - 2\sqrt{n} \rfloor$, both being equal to the corresponding known values for a circular disk. Moreover, in the former case, we characterize all extremal configurations, while for the latter problem we characterize all maximal contact configurations for smooth Radon domains. In both instances, all extremal configurations exhibit periodic order. Our proofs employ a characterization of total separability in terms of hemispherical caps, Auerbach bases of finite dimensional real normed spaces, angle measures in real normed planes, isoperimetric polyominoes, properties of Radon curves, and an approximation of smooth strictly convex domains by specially constructed domains, which we call Auerbach domains.
5.1 Separable Hadwiger numbers

We begin with a lemma of Minkowski that is often used to express questions about packings of translates of an arbitrary convex body in terms of the corresponding questions for a packing of translates of an o-symmetric convex body. Given a d-dimensional convex body \( K \), we denote the Minkowski symmetrization of \( K \) by \( K_o \) and define it to be

\[
K_o = \frac{1}{2}(K + (-K)) = \frac{1}{2}(K - K) = \frac{1}{2}\{x - y : x, y \in K\}.
\]

Clearly, \( K_o \) is an o-symmetric d-dimensional convex body. Minkowski [136] showed that if \( x_1 + K \) and \( x_2 + K \) are two translates of a convex body \( K \), then they are non-overlapping (resp., touching) if and only if \( x_1 + K_o \) and \( x_2 + K_o \) are non-overlapping (resp., touching). Thus, if \( K \) is a convex body and \( \mathcal{P} = \{x_1 + K, \ldots, x_n + K\} \) is a packing of translates of \( K \) in \( \mathbb{E}^d \), then \( \mathcal{P}_o = \{x_1 + K_o, \ldots, x_n + K_o\} \) is a packing and vice versa. Moreover, the contact graphs of \( \mathcal{P} \) and \( \mathcal{P}_o \) are identical. Here we prove the following additional property of Minkowski symmetrization.

**Lemma 5.1.1.** Let \( K \) be a convex body and \( \mathcal{P} = \{x_1 + K, \ldots, x_n + K\} \) be a set of translates of \( K \) in \( \mathbb{E}^d \). Then \( \mathcal{P} \) forms a totally separable packing if and only if \( \mathcal{P}_o = \{x_1 + K_o, \ldots, x_n + K_o\} \) is a totally separable packing of translates of \( K_o \).

**Proof.** Clearly, \( x_i + K_o \) is a centrally symmetric convex body with center \( x_i \), for all \( 1 \leq i \leq n \). For simplicity, let \( o \in \text{int} K \). Then \( x_i \in x_i + \text{int} K \), for all \( 1 \leq i \leq n \).

Assume that the packing \( \mathcal{P} \) is totally separable. Then for any distinct \( x_p + K, x_q + K \in \mathcal{P} \), there exists a hyperplane \( T \) of \( \mathbb{E}^d \) that separates \( x_p + K \) and \( x_q + K \) and is disjoint from the interiors of all packing elements in \( \mathcal{P} \). Thus \( T \) partitions \( \mathcal{P} \) into two subsets, each containing the elements of \( \mathcal{P} \) that lie in one closed half-space of \( \mathbb{E}^d \) bounded by \( T \). Arbitrarily call one of these closed half-spaces the left of \( T \), and the other the right of \( T \). Let \( x_i + K \in \mathcal{P} \) be to the left of \( T \) that is closest to \( T \) (in the norm \( \|\cdot\| \)) and \( x_j + K \in \mathcal{P} \) be to the right of \( T \) that is closest to \( T \), breaking ties arbitrarily. Let \( L \) be the line orthogonal to \( T \) and passing
Figure 5.1: Lemma 5.1.1 – the totally separable form of Minkowski’s lemma.

through \( \mathbf{o} \). We project the packings \( \mathcal{P} \) and \( \mathcal{P}_o \) on to \( L \) using the orthogonal projection map 
\( \pi : \mathbb{E}^d \rightarrow L \).

We refer to Figure 5.1. Let \([\mathbf{a}_i, \mathbf{b}_i] = \pi(\mathbf{x}_i + \mathbf{K})\) and \([\mathbf{a}_j, \mathbf{b}_j] = \pi(\mathbf{x}_j + \mathbf{K})\). Also let \( \mathbf{c}_i = \pi(\mathbf{x}_i) \) and \( \mathbf{c}_j = \pi(\mathbf{x}_j) \). Finally, if \( \mathbf{d}_i = \mathbf{c}_i - \frac{1}{2}(\mathbf{b}_i - \mathbf{a}_i) \), \( \mathbf{d}_j = \mathbf{c}_j - \frac{1}{2}(\mathbf{b}_j - \mathbf{a}_j) \), \( \mathbf{e}_i = \mathbf{c}_i + \frac{1}{2}(\mathbf{b}_i - \mathbf{a}_i) \), \( \mathbf{e}_j = \mathbf{c}_j + \frac{1}{2}(\mathbf{b}_j - \mathbf{a}_j) \), then \([\mathbf{d}_i, \mathbf{e}_i] = \pi(\mathbf{x}_i + \mathbf{K}_o)\) and \([\mathbf{d}_j, \mathbf{e}_j] = \pi(\mathbf{x}_j + \mathbf{K}_o)\). Let \( \|\mathbf{b}_i - \mathbf{a}_i\| = \|\mathbf{b}_j - \mathbf{a}_j\| = 2w \). Then \( \|\mathbf{c}_j - \mathbf{c}_i\| \geq 2w \) and therefore, the closed line segments \([\mathbf{d}_i, \mathbf{e}_i]\) and \([\mathbf{d}_j, \mathbf{e}_j]\) do not overlap. Thus, there exists a translate \( T_o \) of \( T \) that separates \( \mathbf{x}_i + \mathbf{K}_o \) and \( \mathbf{x}_j + \mathbf{K}_o \) and is disjoint from the interiors of all packing elements in \( \mathcal{P}_o \). Finally, \( T_o \) separates \( \mathbf{x}_p + \mathbf{K}_o \) and \( \mathbf{x}_q + \mathbf{K}_o \).
Conversely, assume that the packing \( \mathcal{P}_o \) is totally separable. Then there exists a hyperplane \( T_o \) of \( \mathbb{E}^d \) that separates \( x_p + K_o \) and \( x_q + K_o \) and is disjoint from the interiors of all packing elements in \( \mathcal{P}_o \). Arguing on similar lines as above, let \( x_i + K_o \) be an element of \( \mathcal{P}_o \) to the left of \( T_o \) that is closest to \( T_o \) and \( x_j + K_o \) be an element of \( \mathcal{P}_o \) to the right of \( T_o \) that is closest to \( T_o \). Then by arguing as above, we can easily show that there is a translate \( T \) of \( T_o \) that separates \( x_i + K \) and \( x_j + K \) and is disjoint from the interiors of all packing elements in \( \mathcal{P} \). Moreover, it also separates \( x_p + K \) and \( x_q + K \).

We now define a totally separable analogue of the Hadwiger number, which we call the separable Hadwiger number.

**Definition 5.1.2.** Let \( K \) be a convex body in \( \mathbb{E}^d \). We define the separable Hadwiger number \( H_{sep}(K) \) of \( K \) as the maximum number of translates of \( K \) that all touch \( K \) and, together with \( K \), form a totally separable packing in \( \mathbb{E}^d \).

Although the above definition is very natural, it is not very helpful in determining the exact values or good estimates of \( H_{sep}(K) \). For smooth \( o \)-symmetric convex bodies we will find it advantageous to use an alternative but equivalent characterization. We begin by defining what we mean by a hemispherical cap on the boundary of a smooth \( o \)-symmetric convex body.

**Definition 5.1.3.** Let \( K_o \) be a smooth \( o \)-symmetric convex body in \( \mathbb{E}^d \). Let \( p \in \text{bd} \, K_o \) and \( T_p \) denote the unique supporting hyperplane of \( K_o \) at \( p \). Let \( T_p' \) be the hyperplane parallel to \( T_p \) passing through the origin and let \( P_p \) be the half-open plank of \( \mathbb{E}^d \) bounded by \( T_p \) and \( T_p' \), but excluding \( T_p' \). We say that \( C(p) := \text{bd} \, K_o \cap P_p \) is the hemispherical cap (or simply the cap) on \( \text{bd} \, K_o \) centered at \( p \). We define the boundary of the cap \( C(p) \) as \( \text{bd} \, K_o \cap T_p' \) and denote it by \( \partial C(p) \).

Figure 5.2 illustrates Definition 5.1.3 for a convex domain. The reason that we are defining caps only for smooth convex bodies is obvious. We wish to avoid situations where multiple hyperplanes support \( K_o \) at a single boundary point. Also \( o \)-symmetry is used in
Figure 5.2: The cap $C(p)$ on a convex domain $K_o$ (a circular disk in this case) centered at $p$, its boundary $\partial C(p)$ and a separable point set $\{p, q\}$ on $bd K_o$.

defining the caps. We now introduce the notion of a separable point set on the boundary of a smooth $o$-symmetric convex body $K_o$.

**Definition 5.1.4.** Let $K_o$ be a smooth $o$-symmetric convex body in $\mathbb{E}^d$. A set $S \subseteq bd K_o$ is called a separable point set if $x \notin C(y)$ and $y \notin C(x)$, for any $x, y \in S$.

We now characterize total separability of translative packings of a smooth convex body $K_o$ in terms of separable point sets defined on $bd K_o$.

**Lemma 5.1.5.** Let $K_o$ be a smooth $o$-symmetric convex body in $\mathbb{E}^d$. Let $p_1, p_2 \in bd K_o$ and let $K_1 = 2p_1 + K_o$ and $K_2 = 2p_2 + K_o$. Then $K_o, K_1$ and $K_2$ form a totally separable packing if and only if $p_1 \notin C(p_2)$ and $p_2 \notin C(p_1)$.

**Proof.** Let $T_1$ be the unique hyperplane in $\mathbb{E}^d$ supporting $K_o$ (and $K_1$) at the point $p_1$ and let $T_1'$ be the hyperplane parallel to $T_1$ passing through $o$. Note that $T_1$ is also the unique hyperplane in $\mathbb{E}^d$ separating $K_o$ and $K_1$. Similarly, let $T_2$ be the unique hyperplane supporting $K_o$ (and $K_2$) at $p_2$ and $T_2'$ be the parallel hyperplane passing through $o$. For any
hyperplane $T$ of $\mathbb{E}^d$, we write $T^+$ and $T^-$ to denote the two closed half-spaces of $\mathbb{E}^d$ bounded by $T$. Let $K_i \subseteq T_i^+$, $i = 1, 2$.

(\Rightarrow) If $p_2 \notin C(p_1)$, then it is easy to observe that $K_2 \subseteq T_1^-$. Thus, $T_1$ separates $K_1$ from both $K_2$ and $K_o$. Similarly, if $p_1 \notin C(p_2)$, then $K_1 \subseteq T_2^-$ and $T_2$ separates $K_2$ from both $K_1$ and $K_o$.

(\Leftarrow) Now suppose that $K_o$, $K_1$ and $K_2$ form a totally separable packing. Let us also assume that $K_2 \subseteq T_1^+$. By the total separability of packing $\{K_o, K_1, K_2\}$, we must have $p_2 \in T_1$, and so $T_1$ does not separate $K_1$ and $K_2$. If $p_1 = p_2$, then any hyperplane $L \neq T_1$ separating $K_1$ and $K_2$ must pass through $p_1$. Again by the total separability assumption, $L \cap \text{int} K_o = \emptyset$. Therefore, $L$ is a supporting hyperplane of $K_o$ at $p_1$, which contradicts the smoothness of $K_o$. Thus $p_1 \neq p_2$ and the (non-trivial) line segment $[p_1, p_2]$ lies on $T_1 \cap \text{bd} K_o$ as shown in Figure 5.3. By the o-symmetry of $K_o$, $[-p_1, -p_2] \in \text{bd} K_o$ and the parallelogram $P$ with vertices $p_1, p_2, -p_1$ and $-p_2$ satisfies $\text{relint} P \subseteq \text{int} K_o$, where $\text{relint}(S)$ denotes the relative interior of a set $S$. Also, by the symmetry of packing $\{K_o, K_1\}$
about \( p_1 \) and of \( \{K_o, K_2\} \) about \( p_2 \), we have \( P_1 = 2p_1 + P \subseteq K_1 \) and \( P_2 = 2p_2 + P \subseteq K_2 \).

Let \( \mathbb{E}^2 := \text{span}\{p_1, p_2\} \). Then for any hyperplane \( L \neq T_1 \) separating \( K_1 \) and \( K_2 \), the line \( L \cap \mathbb{E}^2 \) separates \( P_1 \) and \( P_2 \) in \( \mathbb{E}^2 \). Since \( L \) cannot intersect relint\( P \), it either passes through \( p_1 \) or \( p_2 \) (but not both). This implies that \( L \) supports \( K_o \) at \( p_1 \) or at \( p_2 \), a contradiction. Thus \( K_2 \subseteq T_1^{-} \) and clearly, \( p_2 \not\in C(p_1) \). Similarly, it can be shown that \( p_1 \not\in C(p_2) \).

The following is an immediate consequence of Lemma 5.1.5 and the fact that in any totally separable packing involving \( K_o, K_1, K_2 \) and possibly other translates of \( K_o \) all touching \( K_o \) as defined above, \( T_1 \) is the unique hyperplane of \( \mathbb{E}^d \) separating \( K_1 \) from all other packing elements.

**Corollary 5.1.6.** If \( K_o \) is a smooth \( o \)-symmetric convex body in \( \mathbb{E}^d \), then \( H_{\text{sep}}(K_o) \) equals the maximum cardinality of a separable point set on \( \text{bd} K_o \).

**Definition 5.1.7.** Let \( K_o \) be an \( o \)-symmetric convex body in \( \mathbb{E}^d, d \geq 2 \). A non-zero vector \( x \) in \( (\mathbb{R}^d, \|\cdot\|_{K_o}) \) is said to be *Birkhoff orthogonal* to a non-zero vector \( y \) if \( \|x\|_{K_o} \leq \|x + ty\|_{K_o} \), for all \( t \in \mathbb{R} \) [37]. We denote this by \( x \vdash_{K_o} y \).

Note that in general, Birkhoff orthogonality is a non-symmetric relation, that is \( x \vdash_{K_o} y \) does not imply \( y \vdash_{K_o} x \).

**Definition 5.1.8.** Let \( K_o \) be an \( o \)-symmetric convex body in \( \mathbb{E}^d, d \geq 2 \), and let \( \{e_i : i = 1, \ldots, d\} \) be a basis for \( \mathbb{R}^d \). We call this an *Auerbach basis* of \( (\mathbb{R}^d, \|\cdot\|_{K_o}) \) provided that for every \( i \), \( \|e_i\|_{K_o} = 1 \) and \( e_i \) is Birkhoff orthogonal to every element of the linear subspace \( \text{span}\{e_j : j \neq i, j = 1, \ldots, d\} \) (see [145]).

Plichko [145] showed that for every \( o \)-symmetric \( d \)-dimensional convex body \( K_o \), the normed linear space \( (\mathbb{R}^d, \|\cdot\|_{K_o}) \) possesses at least two Auerbach bases – one corresponding to the centers of the facets of the affine \( d \)-cube of minimum volume circumscribing \( K_o \), and the other corresponding to the vertices of the affine \( d \)-cross polytope of maximum volume inscribed in \( K_o \). Moreover, if these two bases coincide, then \( (\mathbb{R}^d, \|\cdot\|_{K_o}) \) possesses infinitely many Auerbach bases.
Remark 5.1.9. Suppose $K_o$ is a smooth o-symmetric convex body in $\mathbb{E}^d$ and $x, y \in \text{bd } K_o$ with $x \dashv_{K_o} y$. If $T$ is the hyperplane supporting $K_o$ at $x$ and $T'$ is the hyperplane passing through $o$ and parallel to $T$, then clearly $y \in T'$. Conversely, if $x, y \in \text{bd } K_o$ and $y \in T'$, then $x \dashv_{K_o} y$. It follows that $x \dashv_{K_o} y$ if and only if $y \in \partial C(x)$.

We now consider the problem of determining $H_{\text{sep}}(K_o)$ for a smooth o-symmetric convex body, that is, the maximal cardinality of a separable point set on $\text{bd } K_o$. The following lower bound is immediate.

**Lemma 5.1.10.** If $K$ is a convex body in $\mathbb{E}^d$, $d \geq 2$, then

$$H_{\text{sep}}(K) \geq 2d.$$  \hspace{1cm} (5.1)

**Proof.** Let $P$ be the affine $d$-cube of minimal volume circumscribing $K_o$. Reflecting $K_o$ across the facets of $P$ gives $2d$ translates of $K_o$, which along with $K_o$ form a totally separable packing. \qed

We now prove that the lower bound given in Lemma 5.1.10 becomes exact for smooth convex domains. In order to show this we require an auxiliary lemma that deals with an important property of any large enough separable point set on $\text{bd } K_o$, where $K_o$ is a smooth o-symmetric convex domain in $\mathbb{E}^2$.

**Lemma 5.1.11.** Let $K_o$ be a smooth o-symmetric convex domain in $\mathbb{E}^2$ and $S$ be any separable point set consisting of 3 or more points on $\text{bd } K_o$. Then $o \in \text{conv}(S)$, where $\text{conv}(S)$ stands for the convex hull of $S$.

**Proof.** Suppose to the contrary that this is not the case. Then there exists a separable point set $S$ on $\text{bd } K_o$ with $x, y, z \in S$ and $o \notin \text{conv}\{x, y, z\}$. Orienting the boundary of $K_o$ in an arbitrary direction, say counterclockwise, gives an ordering of these three points, say $x, y, z$, such that the line segment $[x, z]$ intersects the line segment $[o, y]$ as shown in Figure 5.4. (Note that the point of intersection could be any point on the half-open line segment $(o, y]$.)
Figure 5.4: In Lemma 5.1.11, (a) three points in a separable point set on \( \text{bd} \, K_o \) and the situation with (b) \( x, z \in \partial C(y) \) or (c) \( x \notin \partial C(y) \).

Let \( T \) be the unique line supporting \( K_o \) at \( y \) and \( T' \) the line parallel to \( T \) passing through \( o \). By assumption, \( x, z \notin C(y) \). If \( x, z \in \partial C(y) \), then \( o \in \text{conv}\{x, z\} \), a contradiction. If either \( x \notin \partial C(y) \) or \( z \notin \partial C(y) \), then \([x, z] \) is disjoint from \([o, y]\), again a contradiction. \( \square \)

We can now compute the separable Hadwiger number of smooth \( o \)-symmetric convex domains exactly. Based on Corollary 5.1.6, the second part of Theorem 5.1.12 proves part (C) of Theorem 1.2.4.

**Theorem 5.1.12.** If \( K_o \) is any smooth \( o \)-symmetric convex domain in \( \mathbb{E}^2 \), then

\[
H_{\text{sep}}(K_o) = 4. \tag{5.2}
\]

Moreover, a separable point set \( S \) on \( \text{bd} \, K_o \) has maximal cardinality if and only if \( S \), when considered as a set of vectors, consists of elements of an Auerbach basis of \((\mathbb{R}^2, \|\cdot\|_{K_o})\) and their antipodes.

**Proof.** First, we prove (5.2) using Corollary 5.1.6. Suppose \( S \) is a maximal cardinality
Two cases arise.

If $k = 1$, then $o$ lies on a line segment joining a pair of antipodal points, say $p$ and $-p$, in $S$. Since $K_\circ$ is smooth and $o$-symmetric, the unique lines tangent to $K_\circ$ at $p$ and $-p$ are parallel. Hence $bd K_\circ \setminus (C(p) \cup C(-p))$ consists of a pair of antipodal points and so, $|S| \leq 4$. The result follows by Lemma 5.1.10.

If $k = 2$, then we deal with two subcases. Let $V(\Delta^2) = \{p, q, r\}$ be the set of vertices of $\Delta^2$. Suppose that the tangent lines to two of the vertices of $\Delta^2$, say $p$ and $q$, are parallel. Then once again $bd K_\circ \setminus (C(p) \cup C(q))$ consists of a pair of antipodal points and the result follows.

Now suppose that no two of the vertices of $\Delta^2$ are supported by parallel lines (see Figure 5.5). Let $T_p^+$, $T_q^+$ and $T_r^+$ be the unique lines tangent to $K_\circ$ at $p$, $q$ and $r$ bounding the closed half-spaces $T_p^+$, $T_q^+$, $T_r^+$, $T_q^-$ and $T_r^-$ of $E^d$, respectively, such that $K_\circ \subseteq T_p^+ \cap T_q^+ \cap T_r^+$.
Figure 5.6: An extremal configuration arising in Theorem 5.1.12. Note that \( \{p, q\} \) is an Auerbach basis of \((\mathbb{R}^2, \|\cdot\|_{K_o})\).

and \((- p + T_p^+) \cap (- q + T_q^+) \cap (- r + T_r^+) = \{o\}\). Thus, \(T_p, T_q\) and \(T_r\) form a triangle circumscribing \(K_o\) and

\[
\text{bd} K_o = C(p) \cup C(q) \cup C(r),
\]

showing that \(|S| = |V(\Delta^2)| = 3\). From Lemma 5.1.10, this contradicts the maximality of \(|S|\).

The proof of (5.2) also shows that a maximal cardinality separable point set on \(\text{bd} K_o\) consists of a pair of antipodes \(\{p, -p, q, -q\}\) such that \(p \in \partial C(q),\ q \in \partial C(p)\) and \(p\) and \(q\), when considered as vectors, are linearly independent. Thus, by Remark 5.1.9, \(\ p \dashv_{K_o} q\) and \(q \dashv_{K_o} p\) and therefore, \(\{p, q\}\) is an Auerbach basis of \((\mathbb{R}^2, \|\cdot\|_{K_o})\). Conversely, if \(\{p, q\}\) is an Auerbach basis of \((\mathbb{R}^2, \|\cdot\|_{K_o})\), then by Remark 5.1.9, \(\{p, -p, q, -q\}\) is a separable point set on \(\text{bd} K_o\), and by (5.2), has maximal cardinality. This completes the proof.

Figure 5.6 demonstrates a separable point set of maximum cardinality on the boundary of a smooth \(o\)-symmetric convex domain. We now refer to Lemma 5.1.1. Despite using
Figure 5.7: Totally separable lattice packings of affine squares and inscribed affine regular convex hexagons.

-o-symmetry throughout most of this section, Lemma 5.1.1 shows that this assumption can be removed from the statement of Theorem 5.1.12.

**Corollary 5.1.13.** If $K$ is a smooth convex domain in $\mathbb{E}^2$, then $H_{sep}(K) = 4$.

This proves part (A) of Theorem 1.2.4. Of course, there are examples of convex domains with higher separable Hadwiger numbers. Grünbaum [81] showed that the Hadwiger number of an affine square is 8, while that of any other convex domain is 6. Using this and Figure 5.7, it is easy to observe that the separable Hadwiger number of an affine square is 8, while that of an affine regular convex hexagon is 6. Moreover, Lemma 5.1.10 gives $H_{sep}(K) \geq 4$, for any convex domain $K$. Therefore, it follows that for any convex domain $K$, $H_{sep}(K) \in \{4, 5, 6, 8\}$.

### 5.2 Maximum separable contact numbers

In the classical paper [89], Harborth showed that in the Euclidean plane

$$c(n, 2) = \lfloor 3n - \sqrt{12n - 3} \rfloor,$$  \hspace{1cm} (5.3)

for all $n \geq 2$. Brass [55] extended (5.3) showing that if $K$ is a convex domain different from an affine square in $\mathbb{E}^2$ then for all $n \geq 2$, one has $c(K, n, 2) = \lfloor 3n - \sqrt{12n - 3} \rfloor$ and if $K$ is
an affine square, then \( c(K, n, 2) = \lfloor 4n - \sqrt{28n - 12} \rfloor \) for all \( n \geq 2 \).

**Definition 5.2.1.** Let \( K \) be a convex body in \( \mathbb{E}^d \) and \( n \) be a positive integer. We define the *maximum separable contact number* \( c_{\text{sep}}(K, n, d) \) of \( K \) as the maximum number of edges in a contact graph of a totally separable packing of \( n \) translates of \( K \) in \( \mathbb{E}^d \).

The following natural question was raised in [29].

**Problem 5.2.2.** Find an analogue of Brass’ result for totally separable translative packings of an arbitrary convex domain \( K \).

Recently, Bezdek, Szalkai and Szalkai [36] used Harborth’s method to solve the Euclidean case of Problem 5.2.2; namely, they proved that

\[
c_{\text{sep}}(n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor.
\] (5.4)

In this section, we address Problem 5.2.2 for every smooth strictly convex domain \( K \), showing that \( c_{\text{sep}}(K, n, 2) = c_{\text{sep}}(n, 2) \), which proves part (B) of Theorem 1.2.4.

### 5.2.1 Preliminaries on angle measures and Radon planes

**Definition 5.2.3.** Let \( K_o \subseteq \mathbb{R}^2 \) be an \( o \)-symmetric convex domain in \( \mathbb{E}^2 \). An *angular measure*, also called an *angle measure*, in \((\mathbb{R}^2, \|\cdot\|_{K_o})\) is a measure \( \mu \) defined on \( \text{bd}K_o \) that can be extended in a translation-invariant way to measure angles anywhere and satisfies the following properties [55].

(i) \( \mu(\text{bd}K_o) = 2\pi \).

(ii) For any Borel set \( X \subseteq \text{bd}K_o, \mu(X) = \mu(-X) \).

(iii) For each \( x \in \text{bd}K_o, \mu(\{x\}) = 0 \).

For any \( x, y \in \text{bd}K_o \), we write \( \mu([x, y]_{K_o}) \) for the measure of the angle subtended by the arc \([x, y]_{K_o} \) at \( o \). In [6, 68], angle measures are required to satisfy a fourth non-degeneracy
condition, namely, for any \( x \neq y \in \text{bd} K_o \), \( \mu([x,y]_{K_o}) > 0 \). Here it suffices to adopt Brass’
definition. We refer the interested reader to [6] for a very recent expository treatment of angle measures.

Note that the usual Euclidean angle measure in the plane satisfies these conditions. Moreover for any angle measure in \((\mathbb{R}^2, \|\cdot\|_{K_o})\), the sum of interior angles of any simple \( n \)-gon in \( \mathbb{R}^2 \) equals \((n - 2)\pi\) [55]. This observation will be used in the proof of Theorem 5.2.13. We now define some concepts that play an important role in the sequel.

**Definition 5.2.4.** An angular measure \( \mu \) in the plane \((\mathbb{R}^2, \|\cdot\|_{K_o})\) is called a **B-measure** [72] if for any \( x, y \in \text{bd} K_o \), \( x \perp_{K_o} y \) implies that \( \mu([x,y]_{K_o}) = \pi/2 \).

**Definition 5.2.5.** Given an \( o \)-symmetric convex domain \( K_o \) in \( \mathbb{E}^2 \), the normed plane \((\mathbb{R}^2, \|\cdot\|_{K_o})\) is called a **Radon plane** if for any \( x, y \in \text{bd} K_o \), \( x \perp_{K_o} y \) implies \( y \perp_{K_o} x \) (see, for example, [7, 129]). In other words, a Radon plane is one in which the relation of Birkhoff orthogonality is symmetric. We define a **Radon domain** as the closed unit disk of a Radon plane.

**Definition 5.2.6.** If \( K_o \) is an \( o \)-symmetric convex domain in \( \mathbb{E}^2 \) that is not necessarily a Radon domain, then an (non-trivial) arc \( a \subseteq \text{bd} K_o \) is said to be a **Radon arc** if \( x \perp_{K_o} y \) for any \( x \in a \) and \( y \in \text{bd} K_o \) implies \( y \perp_{K_o} x \) [72].

FankhANEL [72] showed that if the boundary of an \( o \)-symmetric convex domain \( K_o \) contains a Radon arc, then \((\mathbb{R}^2, \|\cdot\|_{K_o})\) possesses a **B-measure**. Moreover, if \( K_o \) is a smooth Radon domain, then \((\mathbb{R}^2, \|\cdot\|_{K_o})\) possesses a strictly increasing **B-measure**. Shortly, we will construct a very concrete example of a **B-measure**.

### 5.2.2 Smooth **B-domains** and their maximum separable contact numbers

In this section, we define a class of convex domains, which we call **B-domains**, and obtain an exact formula for the maximum contact number of totally separable packings of \( n \) translates
of any smooth $B$-domain. The name $B$-domain stems from the connection with $B$-measures.

**Definition 5.2.7.** Let $D \subseteq \mathbb{R}^2$ be an $o$-symmetric convex domain. Then $D$ is called a $B$-domain if there is a $B$-measure defined in $(\mathbb{R}^2, \|\cdot\|_D)$.

From [72], the class of $B$-domains includes the circular disk, Radon domains including the affine regular convex hexagon (in fact, all regular convex $2n$-gons where $n$ is odd [129]) and, more generally, any convex domain whose boundary contains a Radon arc. We will shortly see how having a $B$-measure helps when computing the maximum separable contact number.

Before stating the main result of this section, we take a detour to introduce some ideas that will be needed in its proof and also in Section 5.3. Consider the two-dimensional integer lattice $\mathbb{Z}^2$, which can also be thought of as an infinite plane tiling array of unit squares called lattice cells. For convenience, we imagine these squares to be centered at the integer points, rather than having their vertices at these points.

**Definition 5.2.8.** Two lattice cells of $\mathbb{Z}^2$ are connected if they share an edge. A polyomino or $n$-omino is a collection of $n$ lattice cells of $\mathbb{Z}^2$ such that from any cell we can reach any other cell through consecutive connected cells.

**Definition 5.2.9.** A packing of congruent unit diameter circular disks centered at the points of $\mathbb{Z}^2$ is called a digital circle packing [29, Section 6]. We denote the maximum contact number of such a packing of $n$ circular disks by $c_Z(n, 2)$.

Recall that $c_{\text{sep}}(n, 2) = c_{\text{sep}}(B^2, n, 2)$. Clearly, every digital circle packing is totally separable and $c_Z(n, 2) \leq c_{\text{sep}}(n, 2)$. Consider a digital packing of $n$ circular disks inscribed in the cells of an $n$-omino. Since each circular disk touches its circumscribing square at the mid-point of each edge and at no other point, it follows that the number of edges shared between the cells of the polyomino equals the contact number of the corresponding digital circle packing.

Through the rest of this chapter, $k$, $\ell$ and $\epsilon$ are integers satisfying $\epsilon \in \{0, 1\}$ and $0 \leq k < \ell + \epsilon$. We note that any positive integer $n$ can be uniquely expressed as $n = \ell(\ell + \epsilon) + k$ (as
Harary and Harborth [88] studied minimum-perimeter $n$-ominoes and Alonso and Cerf [2] characterized these in $\mathbb{Z}^2$. The latter also constructed a special class of minimum-perimeter polyominoes called basic polyominoes. Let $n = \ell(\ell + \epsilon) + k$. A basic $n$-mino in $\mathbb{Z}^2$ is formed by first completing a quasi-square $Q_{\alpha \times \beta}$ (a rectangle whose dimensions differ by at most 1 unit) of dimensions $\alpha \times \beta$ with $\{\alpha, \beta\} = \{\ell, \ell + \epsilon\}$ and then attaching a strip $S_{1 \times k}$ of dimensions $1 \times k$ (resp. $S_{k \times 1}$ of dimensions $k \times 1$) to a vertical side of the quasi-square (resp. a horizontal side of the quasi-square). Here, we assume that the first dimension is along the horizontal direction. Moreover, we denote any of the resulting polyominoes by $Q_{\alpha \times \beta} + S_{1 \times k}$ (resp. $Q_{\alpha \times \beta} + S_{k \times 1}$). The results from [2, 88] indirectly show that $c_{Z}(n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor$, which together with (5.4) implies that $c_{sep}(n, 2) = c_{Z}(n, 2)$. Thus, maximal contact digital packings of $n$ circular disks are among the maximal contact totally separable packings of $n$ circular disks.

In order to make use of these ideas, we present analogues of polyominoes and digital circle packings in arbitrary normed planes.

**Definition 5.2.10.** Let $K_0$ be a smooth o-symmetric convex domain in $\mathbb{E}^2$ and $P$ any parallelogram (not necessarily of minimum area) circumscribing $K_0$ such that $K_0$ touches each side of $P$ at its midpoint (and not at the corners of $P$ as $K_0$ is smooth). Let $x$ and $y$ be the midpoints of any two adjacent sides of $P$. Then $-x$ and $-y$ are also points of $\text{bd}K_0 \cap \text{bd}P$. It is easy to see that $\{x, y\}$ is an Auerbach basis of the normed plane $(\mathbb{R}^2, \|\cdot\|_{K_0})$. We call the lattice $L_P$ in $(\mathbb{R}^2, \|\cdot\|_{K_0})$ with fundamental cell $P$, an Auerbach lattice of $K_0$ as we can think of $L_P$ as being generated by the Auerbach basis $\{x, y\}$ of $(\mathbb{R}^2, \|\cdot\|_{K_0})$.

On the other hand, any Auerbach basis $\{x, y\}$ of a smooth o-symmetric convex domain $K_0$ generates an Auerbach lattice $L_P$ of $K_0$, with the fundamental cell determined by the unique lines supporting $K_0$ at $x, y, -x$ and $-y$, respectively. In the sequel, we will use $L_P$ to denote the tiling of $\mathbb{R}^2$ by translates of $P$ as well as the set of centers of the tiling cells.
Indeed, the integer lattice $\mathbb{Z}^2$ is an Auerbach lattice of the circular disk $B^2$.

Given an Auerbach lattice $\mathcal{L}_P$ of a smooth $o$-symmetric convex domain $K_o \subseteq \mathbb{E}^2$ corresponding to the Auerbach basis $\{x, y\}$, we define polyominoes in $\mathcal{L}_P$ as in Definition 5.2.8. We also define basic $n$-ominoes in $\mathcal{L}_P$ along the same lines as in $\mathbb{Z}^2$, with the first dimension along $x$, while the second dimension along $y$. The left and right rows of an $\mathcal{L}_P$-polyomino $p$ are defined along the $x$-direction, and the top and bottom rows are defined along the $y$-direction in the natural way. The base-lines of $p$ are the four sides of a minimal area parallelogram containing $p$ and are designated (in a natural way) as the top, bottom, right and left base-lines of $p$. The graph of $p$, denoted by $G(p)$, has a vertex for each cell of $p$, with two vertices adjacent if and only if the corresponding cells share a side. Figure 5.8 shows some basic polyominoes and their graphs in some Auerbach lattice. We refer to the translates of $K_o$ centered at the lattice points of $\mathcal{L}_P$ (inscribed in the cells of $\mathcal{L}_P$) as $\mathcal{L}_P$-translates of $K_o$. Any packing of such translates will be called an $\mathcal{L}_P$-packing of $K_o$. Since $\mathcal{L}_P$ is a linear image of $\mathbb{Z}^2$, the results of [2, 88] also hold for $\mathcal{L}_P$-polyominoes.

**Lemma 5.2.11.** Let $K_o$ be a smooth $o$-symmetric convex domain, and $p$ be an $n$-omino in an Auerbach lattice $\mathcal{L}_P$ of $K_o$.

(i) If $\mathcal{P}$ is a packing of $n$ translates of $K_o$ inscribed in the cells of $p$, then $G(p)$ is the contact
graph of $\mathcal{P}$, and therefore the number of edges in $G(p)$ is equal to the contact number $c(\mathcal{P})$ of $\mathcal{P}$.

(ii) If in addition $p$ is a minimum-perimeter (or basic) $n$-omino, then $c(\mathcal{P}) = \lfloor 2n - 2\sqrt{n} \rfloor$.

Proof. Since $K_o$ is smooth, no $L_p$-translate of $K_o$ meets the cell of $L_p$ containing it at a corner of the cell. Also any $L_p$-translates of $K_o$ touches the cell containing it at the midpoints of the four sides of the cell. Therefore, two $L_p$-translates of $K_o$ touch if and only if the cells of $L_p$ containing them share a side. This proves (i).

Statement (ii) now follows from (i) and [2, 88]. \hfill \Box

We now show in Theorem 5.2.13 that $c_{\text{sep}}(D, n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor$ for any smooth $B$-domain $D$. The existence of a $B$-measure plays a key role in the proof, as it provides us with a Euclidean-like angle measure. The proof also heavily relies on the $L_p$-packing ideas discussed above. Smoothness is needed, for it allows us to make use of Lemma 5.2.11 and Theorem 5.1.12, or the following special case of Theorem 5.1.12 that can be proved independently.

Remark 5.2.12. Suppose $D$ is a smooth $B$-domain, $\mu$ a $B$-measure in $(\mathbb{R}^2, \| \cdot \|_D)$ and $S$ any separable point set on $\text{bd} D$. Then for any $x, y \in S$, $\mu([x, y]_D) \geq \pi/2$. Since $\mu(\text{bd } D) = 2\pi$, this implies that $|S| \leq 4$. By Lemma 5.1.10, we conclude that $H_{\text{sep}}(D) = 4$.

Moreover, Theorem 5.2.13 is sharp in the sense that it no longer holds if we remove the smoothness assumption. This can be seen through the totally separable packing of $9$ translates of an affine regular convex hexagon given in Figure 5.7.

Theorem 5.2.13. If $D$ is a smooth $B$-domain in $\mathbb{E}^2$ and $n \geq 2$, then we have

$$c_{\text{sep}}(D, n, 2) = c_{\text{sep}}(n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor.$$ (5.5)

Proof. First, we establish the lower bound whose proof neither uses smoothness nor the $B$-measure. Consider an Auerbach lattice $L_p$ of $D$ corresponding to an Auerbach basis
\{(x, y) \text{ of } (\mathbb{R}^2, \|\cdot\|_D)\}. \text{ Then } L_p = T(\mathbb{Z}^2) \text{ for some linear transformation } T : \mathbb{R}^2 \to \mathbb{R}^2. \text{ Now for any } n \geq 2, \text{ consider a maximal contact digital packing } C \text{ of } n \text{ circular disks, and let } p \text{ be the corresponding polyomino in } \mathbb{Z}^2. \text{ Then } T(p) \text{ is an } L_p\text{-polyomino with } n \text{ cells. Let } P \text{ be the packing of } L_p\text{-translates of } D \text{ inscribed in the cells of } T(p). \text{ Then by Lemma 5.2.11, the contact number of } P \text{ is at least as large as the contact number of } C. \text{ Thus,} 
\begin{align*}
c_{\text{sep}}(D, n, 2) \geq \lfloor 2n - 2\sqrt{n} \rfloor.
\end{align*}

Since (5.5) clearly holds for } n \leq 3, \text{ we prove the reverse inequality by induction on } n, \text{ the number of translates in the packing. Our approach has its origins in the proof outline of (5.4) in [36], but requires more sophisticated tools. For the sake of brevity, we write } c_{\text{sep}}(n) = c_{\text{sep}}(D, n, 2). \text{ Suppose (5.5) is true for totally separable packings of up to } n - 1 \text{ translates of } D. \text{ By Lemma 5.2.11, we may assume that for } j \leq n - 1, \text{ } L_p\text{-packings of } j \text{ translates of } D \text{ inscribed in the cells of a basic polyomino in } L_p \text{ are among the maximal contact totally separable packings of } j \text{ translates of } D. \text{ Let } G \text{ denote the contact graph of a maximal contact totally separable packing } P \text{ of } n \geq 4 \text{ translates of } D. \text{ Since } n \geq 2 \text{ and } c_{\text{sep}}(n - 1) + 1 = \lfloor 2(n - 1) - 2\sqrt{n - 1} \rfloor + 1 \leq \lfloor 2n - 2\sqrt{n} \rfloor, \text{ we can assume, without loss of generality, that every vertex of } G \text{ has degree at least 2.}

Given a vertex } v \text{ of } G, \text{ let } D^v \text{ denote the corresponding translate in } P. \text{ Let } G - v \text{ denote the graph obtained by deleting } v \text{ and all the edges incident to } v \text{ from } G. \text{ Clearly, } G - v \text{ is the contact graph of the packing } P \setminus \{D^v\}. \text{ For a subgraph } H \text{ of } G, \text{ we define } G - H \text{ analogously. By performing an affine transformation, if necessary, we may assume that the fundamental cell } P \text{ of } L_p \text{ is a square. Moreover, we may assume the } x\text{-direction to be horizontal and the } y\text{-direction to be vertical. This readily allows us to designate the left, right, top and bottom sides of a cell (resp. rows of a polyomino).}

We now show that except in one case, } G \text{ must be 2-connected. Suppose } n = 7 \text{ and } G \text{ is isomorphic to the graph } G(p_7) \text{ of the non-basic polyomino } p_7 \text{ shown in Figure 5.9. We note that } G(p_7) \text{ has minimum degree 2, but is not 2-connected. However, this does not cause any issues as } G \text{ has } 8 = \lfloor 2(7) - 2\sqrt{7} \rfloor \text{ edges and therefore, satisfies the desired upper bound of}
Figure 5.9: A 7-omino $p_7$ whose graph $G(p_7)$ has minimum degree 2 but is not 2-connected.

$\lfloor 2n - 2\sqrt{n} \rfloor$ on the contact number of the underlying packing. Moreover, none of the contact graphs arising through the rest of the proof is isomorphic to $G(p_7)$.

Claim 1: If $G$ is not isomorphic to $G(p_7)$, then $G$ is 2-connected.

Suppose $n = 4$. If $u$ is a vertex of $G$ of degree 1, then $\mathcal{P} \setminus \{D^u\}$ is a totally separable packing of 3 translates of $D$ such that $c(\mathcal{P} \setminus \{D^u\}) = 3 > 2 = c_{\text{sep}}(D, 3, 2)$, a contradiction. Therefore, $G$ has minimum degree at least 2. If $v$ is a vertex of $G$ such that $G - v$ is disconnected, then at least one of the connected components of $G - v$ consists of a single vertex $w$. But then $w$ has degree 1 in $G$, a contradiction. Therefore $G$ is 2-connected and we inductively assume 2-connectedness for contact graphs not isomorphic to $G(p_7)$ of up to $n - 1$ translates of $D$.

Now suppose that $n = \ell(\ell + \epsilon) + k > 4$ and there exists a vertex $v$ of $G$ such that $G - v$ is disconnected. Let $C$ be one of the connected components of $G - v$ containing the least positive number of neighbours of $v$. Since by Theorem 5.1.12 (or Remark 5.2.12), $v$ has degree at most 4, we encounter two cases.

Case I: The component $C$ contains one neighbour of $v$.

Sub-case I(a): If $G - C - v$ also contains 1 neighbour of $v$, then we argue as follows. Suppose $C$ contains $n_1$ vertices and so $G - C - v$ consists of $n_2 = n - n_1 - 1$ vertices. Clearly, $n_1$ and $n_2$ are both at least 2 as otherwise $G$ has a vertex of degree 1. For $i = 1, 2$, let
Figure 5.10: Case I in the proof of Theorem 5.2.13 when $G - C - v$ contains (a) 1 neighbour of $v$, (b) 2 or 3 neighbours of $v$.

$n_i = \ell_i (\ell_i + \epsilon_i) + k_i$ with $\epsilon_i \in \{0, 1\}$, $0 \leq k_i < \ell_i + \epsilon_i$.

Let $p_1$ be a realization of $Q_{(\ell_1 + \epsilon_1) \times \ell_1} + S_{1 \times k_1}$ with the strip $S_{1 \times k_1}$, if non-empty, forming the right row of $p_1$ such that the top base-lines of $Q_{(\ell_1 + \epsilon_1) \times \ell_1}$ and $S_{1 \times k_1}$ coincide. Also let $p_2$ be a realization of $Q_{(\ell_2 + \epsilon_2) \times \ell_2} + S_{1 \times k_2}$ with the strip $S_{1 \times k_2}$, if non-empty, forming the left row of $p_2$ such that the bottom base-lines of $Q_{(\ell_2 + \epsilon_2) \times \ell_2}$ and $S_{1 \times k_2}$ coincide. Denote the bottom-right cell of $p_1$ by $c_1$ and the top-right cell of $p_2$ by $c_2$. Now attach a single cell $c$ to the bottom side of $c_1$ and the right side of $c_2$, and let $p$ be the resulting polyomino as shown in Figure 5.10 (a). Since $\ell_1 + \epsilon_1 \geq 2$, in $p$ there exists at least one cell of $p_1$ other than $c_1$ that shares a side with a cell of $p_2$. Now let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the totally separable $\mathcal{L}_p$-packings of translates of $D$ inscribed in the cells of $p_1$ and $p_2$ in $p$, respectively, and $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \{K_c\}$, where $K_c$ is the $\mathcal{L}_p$-translate of $D$ inscribed in the cell $c$. Then using the inductive assumption about basic polyomino packings, $c(\mathcal{P}) \leq c_{\text{sep}}(n_1) + c_{\text{sep}}(n_2) + 2 = c(\mathcal{P}_1) + c(\mathcal{P}_2) + 2 < c(\mathcal{P}_1) + c(\mathcal{P}_2) + 3 = c(\mathcal{P}')$, a contradiction.

Sub-case I(b): If $G - C - v$ contains 2 or 3 neighbours of $v$, then we modify the proof of I(a) as follows. Suppose $C$ contains $n_1$ vertices and so $G - C$ consists of $n_2 = n - n_1$ vertices. Clearly, $n_1 \geq 2$ and $n_2 \geq 3$ as otherwise $G$ has a vertex of degree 1. For $i = 1, 2,$
let $n_i = \ell_i(\ell_i + \epsilon_i) + k_i$ with $\epsilon_i \in \{0, 1\}$, $0 \leq k_i < \ell_i + \epsilon_i$. Note that $n_2$, $\ell_2$, $\epsilon_2$ and $k_2$ do not all have the same values as in Sub-case I(a). Let $p_1$, $p_2$, $c_1$ and $c_2$ be as in the proof of I(a).

If $G - C - v$ contains 2 neighbours of $v$, then attach the bottom side of $c_1$ to the top side of $c_2$ to form a polyomino $p$. Since $\ell_1 + \epsilon_1 \geq 2$ and $\ell_2 + \epsilon_2 \geq 2$, in $p$ there exists at least one cell of $p_1$ other than $c_1$ that shares a side with a cell of $p_2$.

On the other hand, if $G - C - v$ contains 3 neighbours of $v$ in $G$, say $x$, $y$ and $z$, then by the total separability of $P$ and smoothness of $D$, no two of $D^x$, $D^y$ and $D^z$ touch. Since the minimum degree of $G$ is at least two, each of the vertices $x$, $y$ and $z$ have a neighbour in $G - C - v$. Moreover, again by the total separability of $P$ and smoothness of $D$, $x$, $y$ and $z$ cannot have the same vertex of $G - C - v$ as a common neighbour. Therefore, $n_2 \geq 6$ and $\ell_2 + \epsilon_2 \geq 3$. Thus, the top row of $p_2$ consists of at least 3 cells $c_2$, $c$ and $c'$ ordered from right to left. We attach the bottom side of $c_1$ to the top side of $c$ to form a polyomino $p$. Since $\ell_1 + \epsilon_1 \geq 2$, there exists a cell of $p_1$ other than $c_1$ that shares its bottom side with the top side of $c'$.

In either scenario (illustrated in Figure 5.10 (b)), let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the totally separable $L_{\mathcal{P}}$-packings of translates of $D$ inscribed in the cells of $p_1$ and $p_2$ in $p$, and let $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2$. Then again using the inductive assumption about basic polyomino packings, $c(\mathcal{P}) \leq c_{\text{sep}}(n_1) + c_{\text{sep}}(n_2) + 1 = c(\mathcal{P}_1) + c(\mathcal{P}_2) + 1 < c(\mathcal{P}_1) + c(\mathcal{P}_2) + 2 = c(\mathcal{P}')$, a contradiction.

Case II: The component $C$ contains two neighbours of $v$.

In this case, assume that $C$ consists of $n_1 - 1$ vertices, so $G - C$ contains $n_2 = n - n_1 + 1$ vertices. Let $x$ and $y$ be the two neighbours of $v$ in $C$. Then by the total separability of the packing and smoothness of $D$, $D^x$ and $D^y$ do not touch. However, since every vertex in $G$ has degree at least 2, there exists at least one vertex $z \neq x, y$ in $C$. Thus, $n_1 \geq 4$. A similar argument shows that $n_2 \geq 4$ and therefore, $n \geq 7$. Using the degree condition on $v$, the disconnectedness of $G - v$ and the total separability of the underlying packing, we observe that $G(p_7)$ is the only 7-vertex contact graph to which Case II applies. Since, $G$ is not isomorphic to $G(p_7)$, we have $n \geq 8$. For $i = 1, 2$, let $n_i = \ell_i(\ell_i + \epsilon_i) + k_i$ with $\epsilon_i \in \{0, 1\}$,
$0 \leq k_i < \ell_i + \epsilon_i$, and note that $n_1$, $\ell_1$, $\epsilon_1$ and $k_1$ do not all have the same values as in Sub-case I(b). Let $p_1$, $p_2$ and $c_2$ be as in the proof of I(a) and I(b), but let $c_1$ be the bottom-left cell of $p_1$. Since $\ell_1 + \epsilon_1 \geq 2$, there exists a cell $c'_1$ adjacent to $c_1$ in the bottom row of $p_1$. Also since $\ell_2 + \epsilon_2 \geq 2$, there exists a cell $c'_2$ adjacent to $c_2$ in the top row of $p_2$.

As $n \geq 8$, at least one of $p_1$ and $p_2$ consists of at least 5 cells and at least 3 vertical rows. Form an $n$-omino by overlapping the cells $c_1$ and $c_2$ into a single cell $c$. If $p_1$ contains more cells than $p_2$, translate the bottom cell $b$ of the right row $R$ of $p_1$ down and left, and attach it to the bottom side of $c'_1$. Let $R'$ denote the rest of $R$. We move $R'$ so that the bottom side of $R'$ is attached to the top side of $c'_2$. Note that each cell of $R'$ shares its right side with a cell in the left row of $p_1$ as shown in Figure 5.11. If $p_2$ contains more cells than $p_1$, translate the top cell $t$ of the left row $L$ of $p_2$ up and right, and attach it to the top side of $c'_2$. Let $L'$ denote the rest of $L$. We move $L'$ so that the top side of $L'$ is attached to the bottom side of $c'_1$. Note that each cell of $L'$ shares its left side with a cell in the right row of $p_2$ as shown in Figure 5.12.

Let $p$ be the resulting $n$-omino and $\mathcal{P}'$ be the totally separable $L_p$-packing of translates of $D$ inscribed in the cells of $p$. Clearly, $c(\mathcal{P}) + 1 \leq c(\mathcal{P}')$, a contradiction. This proves Claim 1.

Thus, $G$ is a 2-connected planar graph with $c_{sep}(n)$ edges, having minimum vertex degree
at least 2, and so every face of $G$ – including the external one – is bounded by a cycle. Thus, $G$ is bounded by a simple closed polygon $P$. Let $v$ denote the number of vertices of $P$. By Theorem 5.1.12 (or Remark 5.2.12), the degree of each vertex in $G$ is 2, 3 or 4. For $j \in \{2, 3, 4\}$, let $v_j$ be the number of vertices of $P$ of degree $j$. By definition of $B$-domains, there exists a $B$-measure $m$ in $(\mathbb{R}^2, \|\|_D)$ so that using total separability of our packing, the internal angle of $P$ at a vertex of degree $j$ is at least $\frac{(j-1)\pi}{2}$. Since the internal angle sum formula holds for angular measures, the sum of these angles will be $(v - 2)\pi$. Clearly $v = v_2 + v_3 + v_4$, and thus, we get the inequality

$$v_2 + 2v_3 + 3v_4 \leq 2v - 4.$$  \hspace{1cm} (5.6)

Now let $g_j$ be the number of internal faces of $G$ that have $j$ sides. By total separability and smoothness, $j \geq 4$. It follows from Euler’s polyhedral formula that

$$n - c_{\text{sep}}(n) + g_4 + g_5 + \ldots = 1.$$  \hspace{1cm} (5.7)

In the process of adding up the number of sides of the internal faces of $G$, every edge of
$P$ is counted once and all the other edges are counted twice. Therefore,

$$4(g_4 + g_5 + \ldots) \leq 4g_4 + 5g_5 + \ldots = v + 2(c_{\text{sep}}(n) - v). \quad (5.8)$$

This together with (5.7) implies that $4(1 - n + c_{\text{sep}}(n)) \leq v + 2(c_{\text{sep}}(n) - v)$, and thus, we obtain

$$2c_{\text{sep}}(n) - 3n + 4 \leq n - v. \quad (5.9)$$

From $G$, delete the vertices of $P$ along with the edges that are incident to them. From the definition of $c_{\text{sep}}(n - v)$, we get $c_{\text{sep}}(n) - v - (v_3 + 2v_4) \leq c_{\text{sep}}(n - v)$, which together with (5.6) implies that $c_{\text{sep}}(n) \leq c_{\text{sep}}(n - v) + 2v - 4$. By induction hypothesis, $c_{\text{sep}}(n - v) \leq 2(n - v) - 2\sqrt{n - v}$, and so

$$c_{\text{sep}}(n) \leq (2n - 4) - 2\sqrt{n - v}. \quad (5.10)$$

Using (5.9) it follows that $c_{\text{sep}}(n) \leq (2n - 4) - 2\sqrt{2c_{\text{sep}}(n) - 3n + 4}$, and so

$$c_{\text{sep}}(n)^2 - 4nc_{\text{sep}}(n) + (4n^2 - 4n) \geq 0.$$

Finally, since the solutions of the quadratic equation $x^2 - 4nx + (4n^2 - 4n) = 0$ are $x = 2n \pm 2\sqrt{n}$ and $c_{\text{sep}}(n) < 2n$, it follows that $c_{\text{sep}}(n) \leq 2n - 2\sqrt{n}$. \hfill \Box

The arguments given in the proof of the lower bound in Theorem 5.2.13 establish the following lower bound on the maximum separable contact number of $n$ translates of any convex domain. Note that we can drop the assumption of o-symmetry due to Lemma 5.1.1.

**Remark 5.2.14.** For any convex domain $K$, we have $c_{\text{sep}}(K, n, 2) \geq \lfloor 2n - 2\sqrt{n} \rfloor$. 

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5.2.3 **Smooth A-domains and their maximum separable contact numbers**

Before we give the definition of an A-domain, note that although Birkhoff orthogonality is a non-symmetric relation in general, it turns out to be symmetric in Euclidean spaces. That is, for any \( x, y \in \mathbb{E}^d \), \( x \perp_{B^d} y \) if and only if \( y \perp_{B^d} x \).

**Definition 5.2.15.** Let \( A \subseteq \mathbb{E}^2 \) be an \( o \)-symmetric convex domain, \( B \) a circular disk centered at \( o \) and \( c, -c, c', -c' \) non-overlapping arcs on \( \text{bd} B \cap \text{bd} A \) such that for any \( x \in c \) there exists \( x' \in c' \) with \( x \perp_{B} x' \) and vice versa. Then we call \( A \) an Auerbach domain, or simply an A-domain.

Figure 5.13 illustrates Definition 5.2.15. Clearly, for any \( x \in c \) there exist antipodes \( x', -x' \in \text{bd} B \) with \( x \perp_{B} x' \) and \( -x' \). From Definition 5.2.15, we must have \( x' \in c' \) and \( -x' \in -c' \). Moreover, an analogous statement holds for any \( x' \in c' \). Therefore, an A-domain \( A \) can be thought of as an \( o \)-symmetric convex domain in \( \mathbb{E}^2 \) such that \( \text{bd} A \) contains two pairs of antipodal circular arcs all lying on the same circle and with each pair
being Birkhoff orthogonal to the other in $\mathbb{E}^2$. Note that this definition does not exclude the case when more than one set of such arcs occurs on $\text{bd} \, A$, in which case we choose the set of four arcs arbitrarily, or even when $A$ is a circular disk. Another example of an $A$-domain appears in [72, Fig. 2] without the use of the term $A$-domain or any other special name. Given an $A$-domain $A$, we call the four circular arcs $c, -c, c', -c'$ chosen on its boundary, the *circular pieces* of $A$ and write

$$\text{cir}(A) = c \cup (-c) \cup c' \cup (-c').$$

Clearly, if $x, y \in \text{cir}(A)$, then $x \dashv_A y$ if and only if $x \dashv_B y$. We observe that any $x \in \text{cir}(A)$ belongs to an Auerbach basis of $A$. Furthermore, any Auerbach basis of $A$ is either contained in $\text{cir}(A)$ or $\text{bd} \, A \setminus \text{cir}(A)$.

The reason behind defining $A$-domains is two-fold. First, in the next section we prove that any smooth $\sigma$-symmetric strictly convex domain can be approximated arbitrarily closely by an $A$-domain. This leads to an exact computation of the maximum separable contact number for any packing of $n$ translates of a smooth strictly convex domain. Second, one can explicitly construct a $B$-measure on the boundary of each $A$-domain that very closely mirrors the properties of the Euclidean angle measure. We give the construction below.

Let $A$ be an $A$-domain with circular pieces $c, -c, c'$ and $-c'$ lying on the boundary of a circular disk $B$, and let $e$ denote the Euclidean angle measure. Then $e(c) = e(c') = e(-c) = e(-c')$ and we define an angle measure $m$ on $\text{bd} \, A$ as follows. For any arc $a \subseteq \text{bd} \, A$, define

$$m(a) = 2\pi \frac{e(a \cap \text{cir}(A))}{e(\text{cir}(A))}.$$  \hfill (5.11)

Note that $m$ assigns a measure of $\pi/2$ to each of the designated circular pieces on $\text{bd} \, A$ and a measure of 0 to the rest of $\text{bd} \, A$ (including any circular arcs not included among the circular pieces), that is, $m(c) = m(c') = m(-c) = m(-c') = \pi/2$ and $m(\text{bd} \, A \setminus \text{cir}(A)) = 0$.

It is easy to check that $m$ satisfies properties (i-iii) of Definition 5.2.3. Moreover, it
satisfies the following property.

**Lemma 5.2.16.** Let $\textbf{A}$ be an $A$-domain and $m$ the angle measure defined on $\text{bd}\textbf{A}$ as described above. Also let $\textbf{x}, \textbf{y} \in \text{bd}\textbf{A}$ be such that $\textbf{x} \vdash_{\textbf{A}} \textbf{y}$, then

$$m([\textbf{x}, \textbf{y}]_A) = \frac{\pi}{2}.$$  \hspace{1cm} (5.12)

In other words, $m$ is a $B$-measure in $(\mathbb{R}^2, ||\cdot||_A)$ and every $A$-domain is a $B$-domain.

**Proof.** Let $c, -c, c'$ and $-c'$ be the circular pieces of $\textbf{A}$ lying on the boundary of a circular disk $\textbf{B}$. Let $\textbf{x}, \textbf{y} \in \text{bd}\textbf{A}$ be such that $\textbf{x} \vdash_{\textbf{A}} \textbf{y}$. Two cases arise. If $\textbf{x} \in \text{cir}(\textbf{A})$, then $\textbf{x} \vdash_{\textbf{B}} \textbf{y}$ holds. Without loss of generality assume that $\textbf{x} \in c$. Then either $\textbf{y} \in c'$ or $\textbf{y} \in -c'$, and $m([\textbf{x}, \textbf{y}]_A) = m([\textbf{x}, \textbf{y}]_A \cap \text{cir(\textbf{A})}) = m(c) = \pi/2$. If $\textbf{x} \in \text{bd}\textbf{A} \setminus \text{cir(\textbf{A})}$, then necessarily $\textbf{y} \in \text{bd}\textbf{A} \setminus \text{cir(\textbf{A})}$ and $[\textbf{x}, \textbf{y}]_A$ contains exactly one of the circular pieces of $\textbf{A}$, say $c$. Thus, once again, $m([\textbf{x}, \textbf{y}]_A) = m(c) = \pi/2$. $\Box$

As $c_{\text{sep}}(\cdot, n, 2)$ is invariant under affine transformations, we obtain:

**Corollary 5.2.17.** If $\textbf{A}$ is (an affine image of) a smooth $A$-domain in $\mathbb{E}^2$ and $n \geq 2$, we have $c_{\text{sep}}(\textbf{A}, n, 2) = c_{\text{sep}}(n, 2) = \lceil 2n - 2\sqrt{n} \rceil$.

### 5.2.4 Maximum separable contact numbers of smooth strictly convex domains

We begin with the definition of Hausdorff distance between two convex bodies.

**Definition 5.2.18.** Given two (not necessarily $o$-symmetric) $d$-dimensional convex bodies $\textbf{K}$ and $\textbf{L}$, the Hausdorff distance between them is defined as

$$h(\textbf{K}, \textbf{L}) = \inf \{ \epsilon : \textbf{K} \subseteq \textbf{L} + \epsilon \textbf{B}^d, \textbf{L} \subseteq \textbf{K} + \epsilon \textbf{B}^d \}.$$
Figure 5.14: Replacing a part of the boundary of a smooth $\mathbf{o}$-symmetric strictly convex domain $\mathbf{K}_o$ by a circular arc. The circular arc on $\text{bd}\mathbf{B}$ is colored blue, while the green arc represents a part of the boundary of $\mathbf{K}_o$. The red arcs represents the outer and inner boundary of the annulus $\text{bd}\mathbf{K}_o + \epsilon \mathbf{B}^2$. The construction is independent of whether $\mathbf{x}_1 \in \mathbf{K}_o'$ or not.

It is well-known that $h(\cdot, \cdot)$ is a metric on the set of all $d$-dimensional convex bodies [153, page 61]. The first main result of this section shows that we can approximate any smooth $\mathbf{o}$-symmetric strictly convex domain in $\mathbb{E}^2$ arbitrarily closely by an affine image of an $A$-domain with respect to Hausdorff distance. In fact, the following theorem proves a stronger statement.

**Theorem 5.2.19.** Affine images of smooth strictly convex $A$-domains are dense (in the Hausdorff sense) in the space of smooth $\mathbf{o}$-symmetric strictly convex domains. Moreover, given any smooth $\mathbf{o}$-symmetric strictly convex domain $\mathbf{K}_o$, we can construct an affine image $\mathbf{A}'$ of a smooth strictly convex $A$-domain $\mathbf{A}$ such that the length of $\text{bd}\mathbf{A}' \cap \text{bd}\mathbf{K}_o$ can be
made arbitrarily close to the length of $\text{bd} \mathbf{K}_o$.

Proof. Let $\mathbf{K}_o$ be a smooth $\mathbf{o}$-symmetric strictly convex domain, and $\epsilon > 0$ be sufficiently small. We describe the construction of a smooth strictly convex $\mathbf{A}$-domain $\mathbf{A}$ with the property that $h(\mathbf{A}', \mathbf{K}_o) \leq \epsilon$, for some image $\mathbf{A}' = T(\mathbf{A})$ of $\mathbf{A}$ under an invertible linear transformation $T : \mathbb{E}^2 \rightarrow \mathbb{E}^2$. Let $\mathbf{K}'_o := T^{-1}(\mathbf{K}_o)$. We note that $\mathbf{K}'_o$ is a smooth $\mathbf{o}$-symmetric strictly convex domain in $\mathbb{E}^2$. It is sufficient to show that $\text{bd} \mathbf{A}$ lies in the annulus $\text{bd} \mathbf{K}'_o + \epsilon \mathbf{B}^2$ (see Figure 5.14) and the length of $\text{bd} \mathbf{A} \cap \text{bd} \mathbf{K}'_o$ can be made arbitrarily close to the length of $\text{bd} \mathbf{K}'_o$ during the construction.

We choose $T$ so that the minimal area parallelogram $\mathbf{P}$ containing $\mathbf{K}'_o$ is a square. Let $x, y, -x, -y \in \text{bd} \mathbf{K}'_o$ be the midpoints of the sides of $\mathbf{P}$. Then $\{x, y\}$ is an Auerbach basis of $(\mathbb{R}^2, \|\cdot\|_{\mathbf{K}'_o})$ [145] and, by strict convexity, $\text{bd} \mathbf{K}'_o$ intersects $\text{bd} \mathbf{P}$ only at points $x, -x, y$ and $-y$. Let $\mathbf{B}$ be the circular disk centered at $\mathbf{o}$ that touches $\text{bd} \mathbf{P}$ at the points $x, y, -x, -y$. Without loss of generality, we may assume that the side of $\mathbf{P}$ passing through $x$ is horizontal and $y$ lies on the clockwise arc on $\text{bd} \mathbf{K}'_o$ from $x$ to $-x$.

Let $U_\epsilon = \text{bd} \mathbf{K}'_o + \epsilon \mathbf{B}^2$ be the $\epsilon$-annular neighbourhood of $\text{bd} \mathbf{K}'_o$ with outer boundary curve $\partial U^+$ and inner boundary curve $\partial U^-$. Let $x^+$ be the unique point of intersection of $\partial U^+$ and $x + \epsilon \mathbf{B}^2$. Moving clockwise along $\partial U^+$ starting from $x^+$, let $u$ be the first point where $\partial U^+$ intersects $\text{bd} \mathbf{P}$. Starting from $x$ and moving along $\text{bd} \mathbf{K}'_o$ clockwise, choose a point $p \in \text{bd} \mathbf{K}'_o$ so that the tangent line supporting $\mathbf{K}'_o$ at $p$ intersects $\mathbf{x}, u$. This unique point of intersection is represented by $v$ in Figure 5.14. Note that by the strict convexity of $\mathbf{K}'_o$, such a point $p$ necessarily exists (for example, choose the point on $\text{bd} \mathbf{K}'_o$ directly below $u$), and any such $p$ can be replaced by any point on the open arc $(x, p)_{\mathbf{K}'_o}$. Now choose a point $x_1 \in \text{bd} \mathbf{B}$ close to $x$ in the clockwise direction so that the line tangent to $\mathbf{B}$ at $x_1$ intersects $(p, v)$. In Figure 5.14, $q$ denotes this point of intersection. Again note that such a point $x_1$ necessarily exists as the line supporting $\mathbf{B}$ at $x$ is horizontal. Moreover, $x_1$ can be replaced by any point on the open arc $(x, x_1)_{\mathbf{B}}$. Therefore, we may assume that $x_1 \neq q$ and so $p, q$ and $x_1$ form a triangle $\Delta$. Thus there exists a (actually, infinitely many) smooth
Figure 5.15: Construction of a smooth strictly convex $A$-domain approximating the smooth $o$-symmetric strictly convex domain $K'_o$ in the proof of Theorem 5.2.19. The circular arcs are colored blue, while the green arcs represent parts of the boundary of $K_o$. The red arcs represent smooth strictly convex connections.

strictly convex curve $S \subseteq \Delta$ with endpoints $x_1$ and $p$ such that the convex domain

$$A_1 = \text{conv}((\text{bd } K'_o \setminus ([x, p]_{K'_o} \cup [-x, -p]_{K'_o})) \cup S \cup [x, x_1]_B \cup (-S) \cup [-x, -x_1]_B)$$

obtained by replacing the antipodal boundary arcs $[x, p]_{K'_o}$ and $[-x, -p]_{K'_o}$ of $K'_o$ with the antipodal circular arcs $[x, x_1]_B$ and $[-x, -x_1]_B$, and the smooth and strictly convex connecting curves $S$ and $-S$, is a smooth $o$-symmetric strictly convex domain with $\text{bd } A_1 \subseteq \text{bd } K'_o + \epsilon B^2$. Repeat the procedure for $A_1$, but this time move counterclockwise along $\text{bd } A_1$ starting from $x$. The result is another smooth $o$-symmetric strictly convex domain $A_2$ with $\text{bd } A_2 \subseteq \text{bd } K'_o + \epsilon B^2$.

Let $[x_1, x_2]_B \subseteq \text{bd } A_2$ be the counterclockwise circular arc containing $x$ (but not necessarily centered at $x$) obtained in this way. We say that $[x_1, x_2]_B$ is a replacement arc for $K'_o$ at $x$ (and therefore, $[-x_1, -x_2]_B$ is a replacement arc for $K'_o$ at $-x$). Let $y_1, y_2 \in \text{bd } B$ be such that $x_1 \vdash_B y_1$ and $x_2 \vdash_B y_2$ (and so $y_1 \vdash_B x_1$, $y_2 \vdash_B x_2$). By choosing $[x_1, x_2]_B$ small enough, we can ensure that $[y_1, y_2]_B$ is a replacement arc for $K'_o$ at $y$. Let $A$ be the resulting convex domain as illustrated in Figure 5.15. Then $A$ is a smooth strictly convex
A-domain with circular pieces \([x_1, x_2]_B, [-x_1, -x_2]_B, [y_1, y_2]_B\) and \([-y_1, -y_2]_B\). Clearly, \(h(A, K'_o) \leq \epsilon\). Furthermore, we can make the length of \(\text{bd} A \cap K'_o\) as close to the length of \(\text{bd} K'_o\) as we like. Therefore, \(h(A', K_o) \leq \epsilon\) and we can make the length of \(\text{bd} A' \cap K_o\) as close to the length of \(\text{bd} K_o\) as we like. 

In Section 5.1, we found that the separable Hadwiger number remains constant over the class of smooth convex domains. Corollary 5.2.17 shows that the maximum separable contact number of any packing of \(n\) translates of an affine image of a smooth \(A\)-domain is the same as the corresponding number for the circular disk. Secondly, we observed that affine images of smooth strictly convex \(A\)-domains form a dense subset of the set of smooth \(o\)-symmetric strictly convex domains. It is, therefore, natural to ask if the maximum separable contact number also remains constant over the set of all smooth strictly convex domains (as we can drop \(o\)-symmetry due to Lemma 5.1.1). We conclude this section by proving that this is indeed the case.

**Corollary 5.2.20.** Let \(K\) be a smooth strictly convex domain in \(\mathbb{E}^2\) and let \(n \geq 2\). Then \(c_{\text{sep}}(K, n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor\).

**Proof.** By Lemma 5.1.1, it suffices to prove the result for smooth \(o\)-symmetric strictly convex domains. Let \(K_o\) be such a domain. By Remark 5.2.14, the lower bound \(c_{\text{sep}}(K_o, n, 2) \geq \lfloor 2n - 2\sqrt{n} \rfloor\) holds. Let \(\mathcal{P}\) be a maximal contact totally separable packing of \(n\) translates of \(K_o\), and \(\mathcal{H}\) be a finite set of lines in \(\mathbb{E}^2\) disjoint from the interiors of the translates in \(\mathcal{P}\) such that any two translates are separated by at least one line in \(\mathcal{H}\). We will construct a smooth strictly convex \(A\)-domain \(A\) such that \(c_{\text{sep}}(K_o, n, 2) \leq c_{\text{sep}}(A', n, 2)\), for \(A' = T(A)\), where \(T : \mathbb{E}^2 \to \mathbb{E}^2\) is a properly chosen invertible linear transformation.

Let \(K_o + c \in \mathcal{P}\). By strict convexity of \(K_o\), there exist finitely many points \(c_1, \ldots, c_m \in \text{bd}(K_o + c)\) where \(K_o + c\) touches other translates in \(\mathcal{P}\) and the lines in \(\mathcal{H}\). Then we call the points \(c_i - c \in \text{bd} K_o, \ i = 1, \ldots, m\), the contact positions on \(K_o\) corresponding to \(K_o + x \in \mathcal{P}\). Let \(\text{Con}(\mathcal{P})\) denote the set of all contact positions on \(K_o\) corresponding to all
the translates in $\mathcal{P}$.

Let $x, y, P$ and $B$ be as in the proof of Theorem 5.2.19. Using Theorem 5.2.19, construct a smooth strictly convex $A$-domain $A$ with circular pieces $c = [x_1, x_2]_A = [x_1, x_2]_B$, $-c = [-x_1, -x_2]_A = [-x_1, -x_2]_B$, $c' = [y_1, y_2]_A = [y_1, y_2]_B$ and $-c' = [-y_1, -y_2]_A = [-y_1, -y_2]_B$.

Also, using Theorem 5.2.19, we can make $c = [x_1, x_2]_A$ sufficiently small so that

$$\text{Con}(P) \subseteq (\text{bd } A' \cap \text{bd } K_o) \cup \{\pm T(x), \pm T(y)\}.$$  

(Recall that both $B$ and $K'_o = T^{-1}(K_o)$ are supported by the sides of $P$ at the points $\{\pm x, \pm y\}$.) Moreover, by making $c = [x_1, x_2]_A$ sufficiently small we ensure that the arrangement $P' = \{A' + c : K_o + c \in P\}$ forms a packing, and $L \cap (A' + c) = L \cap (K_o + c)$ for all $K_o + c \in P$ and $L \in \mathcal{H}$. Thus $P'$ is a totally separable packing with at least $c_{\text{sep}}(K_o, n, 2)$ contacts.

This proves part (B) of Theorem 1.2.4. We conjecture the following generalization.

**Conjecture 5.2.21.** If $K$ is a smooth convex domain, then $c_{\text{sep}}(K, n, 2) = [2n - 2\sqrt{n}]$.

We note that strict convexity plays an important role in the approximation procedure developed in the proof of Theorem 5.2.19 and used in the proof of Corollary 5.2.20. This indicates that an attempt to prove Conjecture 5.2.21 would probably require a different approach.

## 5.3 Crystallization in smooth Radon planes

The aim of this section is to characterize the totally separable packings having the maximum separable contact numbers given in Theorem 1.2.4. We achieve this for smooth Radon domains showing that these extremal configurations form lattice packings that lie on lattices generated by the Auerbach bases of the underlying normed plane $(\mathbb{R}^2, \|\cdot\|_R)$.
5.3.1 Sticky pair-potentials and the separable contact numbers

Let $K_o$ be a smooth $o$-symmetric convex domain. To illustrate the connection of our results with crystallization, we introduce the ground-state problem for the so-called sticky pair-potential in the normed plane $(\mathbb{R}^2, \|\cdot\|_{K_o})$ given by

$$V(\|x_i - x_j\|_{K_o}) = \begin{cases} +\infty, & \text{if } 0 \leq \|x_i - x_j\|_{K_o} < 2, \\ -1, & \text{if } \|x_i - x_j\|_{K_o} = 2, \\ 0, & \text{if } \|x_i - x_j\|_{K_o} > 2. \end{cases}$$

(5.13)

Suppose $n$ particles interacting under this pair-potential are placed at the points $x_i, i = 1, \ldots, n$, in $\mathbb{R}^2$. Then (at low temperatures) their total energy is given by

$$E(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i,j=1, i \neq j}^n V(\|x_i - x_j\|_{K_o}).$$

(5.14)

The ground-state problem is to characterize the configurations $\{x_i : i = 1, \ldots, n\}$ of particles that minimize total energy and demonstrate that these configurations are crystalline [93, 162].

Note that most of the literature considers this problem in the Euclidean plane only, a case that was solved in [93]. The term ‘sticky potential’ stems from the intuition that we can imagine hard impenetrable unit disks (in the norm) $x_i + K_o$ and $x_j + K_o$, centered at $x_i$ and $x_j$. Then the ground-state problem becomes equivalent to characterizing the packings of $n$ translates of $K_o$ that achieve the maximum contact number $c(K_o, n, 2)$.

Suppose that we now introduce the physical constraint that the packings must be totally separable. In other words, we are interested in the the configurations $\{x_i : i = 1, \ldots, n\}$ such that the packing $\{x_i + K_o : i = 1, \ldots, n\}$ is totally separable and $E(x_1, \ldots, x_n)$ is minimum under this restriction. It is easy to see that these configurations are exactly the ones achieving the maximum separable contact number $c_{\text{sep}}(K_o, n, 2)$. Thus, the results in Section 5.3.3 can be interpreted as crystallization of totally separable ground-states of sticky
5.3.2 Smooth Radon domains and strict convexity

Several papers discuss the construction of Radon curves [7, 37, 65, 128] dating back to Radon’s paper [147] where these domains were first introduced.

The classic example of a Radon domain, given by Day [65], is the unit disk of the plane $(\mathbb{R}^2, \| \cdot \|_{p,q})$, where $1 \leq p, q, \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and

$$
\|(x, y)\|_{p,q} = \begin{cases} 
(x^p + y^p)^{1/p}, & x, y \geq 0 \text{ or } x, y \leq 0, \\
(x^q + y^q)^{1/q}, & \text{otherwise.}
\end{cases}
$$

The case where $p = 1, q = \infty$ gives the regular convex hexagon depicted in Figure 5.7. For $1 < p, q < \infty$, the construction yields a smooth Radon domain. Furthermore, Birkhoff [37] describes a general procedure for generating smooth Radon domains.

We now state an interesting property of Radon domains.

**Proposition 5.3.1.** A Radon domain is smooth if and only if it is strictly convex.

**Proof.** Let $R$ be a smooth Radon domain. Suppose that $\text{bd } R$ contains a line segment, and let $a$ and $b$ be two distinct points on that segment. Let $c \in \text{bd } R$ be a point such that $a \dashv_R c$ and $b \dashv_R c$. Note that such a point $c$ necessarily exists by the construction of a Radon domain. Since Birkhoff orthogonality is symmetric on a Radon domain, $c \dashv_R a$ and $c \dashv_R b$, which would imply that $R$ has two distinct supporting lines at $c$, a contradiction.

Conversely, suppose that $R$ is strictly convex but not smooth. Let $c \in \text{bd } R$ be such that $R$ has two distinct supporting lines through $c$. Then there exist two distinct points $a, b \in R$ with $c \dashv_R a$, $c \dashv_R b$, and $b \neq -a$. Since $a \dashv_R c$ and $b \dashv_R c$, it follows that there are supporting lines at $a$ and $b$ which are parallel. Strict convexity would imply that $b = -a$, a contradiction. \qed
5.3.3 Proof of part (D) in Theorem 1.2.4

A consequence of Corollary 5.2.20 is the following stronger form of part (ii) of Lemma 5.2.11 for smooth $\mathfrak{o}$-symmetric strictly convex domains.

Lemma 5.3.2. Let $K_o$ be a smooth $\mathfrak{o}$-symmetric strictly convex domain, and $p$ an $n$-omino in an Auerbach lattice $L_p$ of $K_o$. If $p$ is a minimum-perimeter (or basic) $n$-omino, then $c(p) = c_{\text{sep}}(K_o, n, 2) = \lceil 2n - 2\sqrt{n} \rceil$. Thus, $L_p$-packings of $K_o$ are among the maximal contact totally separable packings of translates of $K_o$.

We require the following lemma about the contact graphs of totally separable packings of translates of a smooth $\mathfrak{o}$-symmetric strictly convex domain having the maximum contact number.

Lemma 5.3.3. Let $K_o$ be a smooth $\mathfrak{o}$-symmetric strictly convex domain, and let $n = \ell(\ell + \epsilon) + k \geq 4$ be the unique decomposition of $n$ with $k \neq 1$. Let $P$ be a totally separable packing of $n$ translates of $K_o$ with contact graph $G$ and $c(P) = c_{\text{sep}}(K_o, n, 2) = \lceil 2n - 2\sqrt{n} \rceil$. Then $G$ has minimum degree at least 2 and is 2-connected.

Proof. Fix an Auerbach lattice $L_p$ of $K_o$ corresponding to an Auerbach basis $\{x, y\}$. By performing an affine transformation, if necessary, we may assume that the fundamental cell $P$ is a square with the $x$-direction being horizontal and the $y$-direction being vertical. All polyominoes considered in this proof are defined over $L_p$.

Suppose $|P| = 4$. If $u$ is a vertex of $G$ of degree 1, then $P \setminus \{K_o^u\}$ is a totally separable packing of 3 translates of $K_o$ such that $c(P \setminus \{K_o^u\}) = 3 > 2 = c_{\text{sep}}(K_o, 3, 2)$, a contradiction. Therefore $G$ has minimum degree at least 2. Now suppose $v$ is a vertex of $G$ such that $G - v$ is disconnected. Then at least one of the connected components of $G$ consists of a single vertex $w$. But then $w$ has degree 1 in $G$, again a contradiction. Therefore, $G$ is 2-connected.

Assume that $n$ is the least positive integer for which there exists a totally separable packing $P$ of $n = \ell(\ell + \epsilon) + k > 4$, $k \neq 1$, translates of $K_o$ with $c_{\text{sep}}(K_o, n, 2)$ contacts such that the contact graph $G$ has minimum degree 1. Let $v$ be a vertex of $G$ with degree 1. Then
c(P \{K^o_o\}) = c_{sep}(K_o, n, 2) - 1 \geq [2(n - 1) - 2\sqrt{n - 1}] = c_{sep}(K_o, n - 1, 2) \geq c(P \{K^o_o\}),

or

\begin{equation}
c(P \{K^o_o\}) = c_{sep}(K_o, n - 1, 2) = c_{sep}(K_o, n, 2) - 1.
\end{equation}

(5.15)

Since \( k \neq 1, \)

\[ n - 1 = \begin{cases} 
\ell(\ell + \epsilon - 1) + (\ell - 1), & \text{if } k = 0, \\
\ell(\ell + \epsilon) + (k - 1), & \text{otherwise.} 
\end{cases} \]

Also \( k - 1 < \ell, \) so in either case, any basic \((n - 1)\)-omino \( p \) consists of a quasi-square \( Q \) with either dimension at least \( \ell, \) and a strip \( S \) of length less than \( \ell \) attached to one of its sides. Therefore, we can always add a cell to \( p \) sharing one side with \( Q \) and one side with \( S. \)

By Lemma 5.2.11 (i) and Lemma 5.3.2,

\[ c_{sep}(K_o, n, 2) \geq c_{sep}(K_o, n - 1, 2) + 2, \]

which contradicts (5.15). Thus, no such \( n \) exists.

We refer back to the proof of Theorem 5.2.13 and note that the part that proves 2-connectedness is independent of the existence of a \( B \)-measure, and therefore can be reused here. Furthermore, the case \( n = 7 \) is excluded here since \( k \neq 1. \)

Now assume that \( n = \ell(\ell + \epsilon) + k > 4, k \neq 1, \) is least such that there exists a maximal contact totally separable translative packing \( P \) of \( K_o \) whose contact graph \( G \) is not 2-connected. Then there exists a vertex \( v \) of \( G \) such that \( G - v \) is disconnected. By the proof of Claim 1 in the proof of Theorem 5.2.13, it follows that no such \( n \) exists.

We now prove Part (D) of Theorem 1.2.4. The proof makes use of Proposition 5.3.1, Lemma 5.3.3 and the fact from [72], stated in Section 5.2.1, that if \( R \) is a smooth Radon domain, then there exists a strictly increasing \( B \)-measure on \( bd R. \) This also means that \( R \) is a \( B \)-domain and Theorem 5.2.13 is applicable to \( R. \)

\textbf{Theorem 5.3.4.} Let \( R \) be a smooth Radon domain and let \( n = \ell(\ell + \epsilon) + k \geq 4 \) be the
decomposition of a positive integer \( n \) such that \( k \neq 1 \). If \( \mathcal{P} \) is a totally separable packing of \( n \) translates of \( \mathbf{R} \) with \( c_{\text{sep}}(\mathbf{R}, n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor \) contacts, then \( \mathcal{P} \) is a finite lattice packing lying on an Auerbach lattice of \( \mathbf{R} \).

**Proof.** Let \( \mu \) be a strictly increasing \( B \)-measure defined in \( (\mathbb{R}^2, \| \cdot \|_{\mathbf{R}}) \). Let \( G \) be the contact graph of \( \mathcal{P} \). Here we consider \( G \) as a geometric graph having the centers of the packing elements as vertices, with edges of length 2 (in the norm \( \| \cdot \|_{\mathbf{R}} \)) joining adjacent vertices. By Lemma 5.3.3, \( G \) has minimum degree at least 2 and is 2-connected. Therefore, every face of \( G \), including the external face, is bounded by a cycle and \( G \) is bounded by a simple closed polygon \( P \). All faces considered from here on are internal faces of \( G \). By the total separability of \( \mathcal{P} \), each face of \( G \) contains at least 4 edges.

Case I: All faces of \( G \) are quadrilaterals.

Let \( F \) be a quadrilateral face of \( G \) with vertices \( v_i, i = 1, 2, 3, 4 \), as shown in Figure 5.16. Since \( \mathbf{R} \) is smooth and strictly convex, the translates \( \mathbf{R}^{v_1} \) and \( \mathbf{R}^{v_2} \) can only be separated by their unique common tangent line \( T_{12} \) at the unique point of contact \( c_{12} \), which must also be the midpoint of the edge \( v_1 v_2 \) of \( G \). One can similarly define \( T_{34}, T_{13}, T_{24}, c_{34}, c_{13} \) and \( c_{24} \). Since \( \mathcal{P} \) is a totally separable packing, \( T_{12} = T_{34} \) and \( T_{13} = T_{24} \).
We write $\mu(v_i, F)$, $i = 1, 2, 3, 4$, to denote the $\mu$-measure of the internal angle of $F$ at the vertex $v_i$. Without loss of generality, let $v_1 = o$. Then $\{c_{12}, c_{13}\}$ is a separable point set on $\text{bd} \, \mathbf{R}$, and so by Lemma 5.1.5, $c_{13} \notin C^{v_1}(c_{12})$, where $C^{v_1}(c_{12})$ denotes the cap centered at $c_{12}$ on $\text{bd} \, \mathbf{R}^{v_1}$. Note that by Remark 5.1.9, if $c_{13} \in \partial C^{v_1}(c_{12})$, then $c_{12} \not\in \mathbf{R} \, c_{13}$; and as $\mu$ is a $B$-measure, we have $\mu(v_1, F) = \pi/2$. If $c_{13} \notin C^{v_1}(c_{12}) \cup \partial C^{v_1}(c_{12})$, then since $\mu$ is strictly increasing, $\mu(v_1, F) > \pi/2$. But, $\sum_{i=1}^{4} \mu(v_i, F) = 2\pi$, implying $\mu(v_i, F) < \pi/2$ for some $i \neq 1$. Therefore, there exist distinct points $c_{\alpha\beta}, c_{\gamma\delta} \in \{c_{12}, c_{34}, c_{13}, c_{24}\}$ such that, on $\text{bd} \, \mathbf{R}^{v_i}$, we have $c_{\beta\gamma} \in C^{v_1}(c_{\alpha\beta})$. By Lemma 5.1.5, this contradicts the total separability of $\mathcal{P}$. Therefore, we must have $c_{13} \in \partial C^{v_1}(c_{12})$, and so $T_{12}$ is parallel to the side $v_1v_3$ of $F$. A similar argument shows that $T_{13}$ is parallel to the side $v_1v_2$ of $F$. Thus, $F$ is a parallelogram.

Moreover, $\{c_{12}, c_{13}\}$ is an Auerbach basis of $((\mathbb{R}^2, \|\cdot\|)$. Let $P$ be the parallelogram with sides parallel to $T_{12}$ and $T_{13}$, that touches $\mathbf{R}^{v_1}$ in the midpoints $c_{12}, c_{13}, -c_{12}, -c_{13}$ of its sides, and let $\mathcal{L}_P$ be the corresponding Auerbach lattice of $\mathbf{R}$. Then the translates $\mathbf{R}^{v_i}, i = 1, 2, 3, 4$, of $\mathbf{R}$ are inscribed in the cells $v_i + P$, $i = 1, 2, 3, 4$, respectively, of $\mathcal{L}_P$. Therefore, the packing $\{\mathbf{R}^{v_i} : i = 1, 2, 3, 4\}$ is an $\mathcal{L}_P$-packing.

If $n = 4$, we are done. If $n > 4$, then $n \geq 6$ and $G$ contains another face $F'$. Since the dual graph $G^*$ of $G$ is connected, $F'$ shares at least one of its edges with another face of $G$, which we can assume to be $F$. Let $v_1v_2$ be the edge shared between $F$ and $F'$. Then by suitably modifying the arguments above, we can easily see that $F'$ is a translate of $F$. Applying this to each face of $G$ shows that $\mathcal{P}$ is an $\mathcal{L}_P$-packing, as desired.

Case II: $G$ has a non-quadrilateral face $F$.

Since this is not possible for $n = 4$, let $n$ be the least positive integer for which there exists a contact graph $G$ of a totally separable packing $\mathcal{P}$ of $n$ translates of $\mathbf{R}$ with contact number $\lceil 2n - 2\sqrt{n} \rceil$ such that $G$ contains a non-quadrilateral face $F$. We refer back to the proof of Theorem 5.2.13 and use the notation $c_{\text{sep}}(n) = c_{\text{sep}}(\mathbf{R}, n, 2) = \lceil 2n - 2\sqrt{n} \rceil$ for any
smooth Radon domain $R$. We show that no such $n$ exists by proving the sharpened versions

\[ 2(c_{\text{sep}}(n) + 1) - 3n + 4 \leq n - v \]  \hfill (5.16)

and

\[ c_{\text{sep}}(n) + 1 \leq (2n - 4) - 2\sqrt{n - v} \]  \hfill (5.17)

of inequalities (5.9) and (5.10), respectively, where $v$ denotes the number of vertices of $P$. This leads to a contradiction as (5.16) and (5.17) together imply $c_{\text{sep}}(n) \leq 2n - 2\sqrt{n - 1}$.

**Proof of (5.16):** Let $g_j$, $j \geq 4$, denote the number of faces of $G$ with $j$ edges. Note that (5.9) is derived from (5.8). Therefore, it suffices to show that

\[ 2 + 4(g_4 + g_5 + g_6 + \cdots) \leq v + 2(c_{\text{sep}}(n) - v). \]  \hfill (5.18)

as this leads to $2 + 4(1 - n + c_{\text{sep}}(n)) \leq v + 2(c_{\text{sep}}(n) - v)$, which upon rearrangement gives (5.16).

Since $4(g_4 + g_5 + g_6 + \cdots) \leq 4g_4 + 5g_5 + 6g_6 + \cdots = v + 2(c_{\text{sep}}(n) - v)$, it suffices to show
that $g_5 = 0$, that is, $F$ has at least 6 edges.

Assume to the contrary that $F$ is a pentagon. Let $v_i, i = 1, \ldots, 5$, be the vertices of $F$ arranged (say) counterclockwise along the 5-cycle bounding $F$. Let $c_{12}, c_{23}, c_{34}, c_{45}$ and $c_{51}$ be the contact points of the translates $R^{v_i}, i = 1, \ldots, 5$, along $F$ and let $T$ be the unique line supporting $R^{v_1}$ and $R^{v_2}$ at the point $c_{12}$.

By the total separability of $P$ and Lemma 5.1.5, $c_{23} \notin C^{v_2}(c_{12})$ on $bd R^{v_2}$ and $c_{51} \notin C^{v_1}(c_{12})$ on $bd R^{v_1}$. If, in addition, $c_{23} \notin \partial C^{v_2}(c_{12})$ and $c_{51} \notin \partial C^{v_1}(c_{12})$ (Figure 5.17 (a)), then neither $R^{v_3}$ nor $R^{v_5}$ touch $T$, and $T$ separates them. Therefore $T$ intersects $int R^{v_4}$, a contradiction. Now assume that $c_{23} \in \partial C^{v_2}(c_{12})$ (Figure 5.17 (b)). Then $R^{v_3}$ touches $T$ at a unique point $t$. If $c_{34} = t$, then $R^{v_4}$ must touch $R^{v_1}$, contradicting the assumption that $F$ is a non-quadrilateral face of $G$. If $c_{34} \neq t$, then $T$ separates $R^{v_4}$ and $R^{v_5}$, and $c_{45}$ lies on $T$. But then $R^{v_5}$ and $R^{v_1}$ cannot touch, contradicting the assumption that $F$ is a face of $G$. Thus, $g_5 = 0$ and (5.16) is proved.

Proof of (5.17): Suppose that $F$ (a non-quadrilateral face of $G$) does not share any vertex with $P$. Note that $G - P$ is the contact graph of the totally separable packing $P' = P \setminus \{R^y :
\( v \) is a vertex of \( P \). Moreover, \( G - P \) contains the non-quadrilateral face \( F \). Therefore, by the choice of \( n \), \( G - P \) has fewer than \( c_{\text{sep}}(n - v) \) edges and

\[
c(\mathcal{P}') + 1 \leq \lfloor 2(n - v) - 2\sqrt{n - v} \rfloor.
\]

Revisiting the arguments from the proof of Theorem 5.2.13 preceding (5.10), we observe that this proves (5.17).

The only remaining case is when \( F \) shares some vertex with \( P \). Here, in order to prove (5.17), we must obtain strict inequality in (5.6), namely,

\[
v_2 + 2v_3 + 3v_4 < 2v - 4,
\] (5.19)

where \( v_j, j = 2, 3, 4 \), denotes the number of vertices on the boundary polygon \( P \) with degree \( j \) in \( G \). Recall that (5.6) was derived using the internal angle sum formula for \( B \)-measures, and the fact that the measure of an internal angle at a vertex \( v \) of degree \( j \) lying on the boundary polygon \( P \) is at least \( \frac{(j-1)\pi}{2} \). In order to prove (5.19), it suffices to show that the internal angle, \( \mu(v, P) \), at some vertex \( v \) of degree \( j \) lying on \( P \) is strictly larger than \( \frac{(j-1)\pi}{2} \). To the contrary assume that no such vertex \( v \) exists.

Let \( v_i, i = 1 \ldots, k \), with \( k \geq 6 \), be the vertices of \( F \) in counterclockwise cyclic order and \( c_{ij}, i, j = 1, \ldots, k \), with \( j = i + 1 \) (mod \( k \), be the unique point where the translates \( R^{v_i} \) and \( R^{v_j} \) touch. Suppose \( F \) shares an edge, say \( v_1v_2 \), with \( P \), and that \( T \) is the unique line supporting \( R^{v_1} \) and \( R^{v_2} \) at \( c_{12} \). By our assumption, \( \mu(v_1, F) = \mu(v_2, F) = \pi/2 \), and therefore \( c_{23} \in \partial C^{v_2}(c_{12}) \) and \( c_{k1} \in \partial C^{v_i}(c_{12}) \) as shown in Figure 5.18 (a). But this implies that \( R^{v_2} \) and \( R^{v_k} \) touch each other at the point \( t = c_{12} + (v_3 - v_2) = c_{12} + (v_k - v_1) \) lying on \( T \). This contradicts the assumption that \( F \) is a non-quadrilateral face. On the other hand, if \( F \) only shares a single vertex \( v_1 \) with \( P \), then both \( v_1 \) and \( v_2 \) have degree 4 and by Theorem 5.1.12, \( \mu(v_1, F) = \mu(v_2, F) = \pi/2 \) as shown in Figure 5.18 (b). This once again contradicts the assumption that \( F \) is a non-quadrilateral face. \( \square \)
Figure 5.19: Two totally separable ground-states of 22 particles under sticky potential in a smooth Radon plane ($\mathbb{R}^2, \|\cdot\|_R$).

Theorem 5.3.4 is applicable if $k \neq 1$ in the decomposition of $n$. For $k = 1$, we prove the following.

**Corollary 5.3.5.** Let $R$ be a smooth Radon domain and let $n = \ell(\ell + \epsilon) + 1 \geq 8$ be the decomposition of a positive integer $n$. If $\mathcal{P}$ is a totally separable packing of $n$ translates of $R$ with $c_{sep}(R, n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor$ contacts, then all but at most one translate in $\mathcal{P}$ form a lattice packing on an Auerbach lattice of $R$.

**Proof.** Let $G$ be the (geometric) contact graph of $\mathcal{P}$. Two possibilities arise.

If $G$ has a vertex $v$ with degree 1, then $\mathcal{P}' = \mathcal{P} \setminus \{R^v\}$ must be a maximal contact totally separable packing of $n - 1 = \ell(\ell + \epsilon)$ translates of $R$. By Theorem 5.3.4, $\mathcal{P}'$ is necessarily an $\mathcal{L}_P$-packing for some Auerbach lattice $\mathcal{L}_P$ of $R$. This proves the result.

Let $G$ have minimum degree at least 2. Then by revisiting the proof of 2-connectedness in Lemma 5.3.3, we note that since $n \geq 8$, $G$ is 2-connected. Therefore, the proof of Theorem 5.3.4 is applicable to $\mathcal{P}$ and $G$, showing that $\mathcal{P}$ is an $\mathcal{L}_P$-packing for some Auerbach lattice $\mathcal{L}_P$ of $R$. □

Considering the connection with sticky potentials discussed in Section 5.3.1, we can now
determine the totally separable ground-states of such potentials in smooth Radon planes. Figure 5.19 illustrates two such ground-states in a smooth Radon plane \((\mathbb{R}^2, \|\cdot\|_\mathbb{R})\). The particles lie on an Auerbach lattice determined by an Auerbach basis \(\{x, y\}\) of \(\mathbb{R}\). The corresponding cells of the lattice form a minimum-perimeter basic polyomino in either case.

**Corollary 5.3.6.** Let \(\mathbb{R}\) be a smooth Radon domain and let \(V(\|x_i - x_j\|_\mathbb{R})\) denote the sticky pair-potential in the normed plane \((\mathbb{R}^2, \|\cdot\|_\mathbb{R})\) as given in (5.13). Then in any totally separable ground-state of \(n \geq 8\) particles under \(V\) at least \(n - 1\) particles lie on (the centres of the cells of) an Auerbach lattice of \(\mathbb{R}\).
Chapter 6

Construction of tournaments with prescribed imbalance sets

We refer back to Chapter 1 for the necessary background and preliminaries. Reid [148] conjectured that any finite nonempty set of non-negative integers is the score set of some tournament and Yao [169] gave a non-constructive proof of Reid’s conjecture using arithmetic arguments. No constructive proof has been found. In this chapter, we investigate a related problem, namely, which sets of integers are imbalance sets of tournaments. We completely solve the Tournament Imbalance Set (TIS) problem (Problem 1.1.4) and also estimate the minimal order of a tournament realizing an imbalance set. Our proofs are constructive and provide an algorithm to realize any imbalance set as a tournament. Along the way, we generalize the well-known Equal-Sum Subsets (ESS) problem to define the Equal-Sum Sequences (ESSeq) problem and show it to be NP-hard. We then give an algorithm that constructs a tournament from its imbalance set thus solving TIS. Among other things, this algorithm uses a procedure that finds all possible solutions to an ESSeq instance. Some of the results appearing in this chapter form part of [107].
6.1 Characterizing odd imbalance sets

Consider a tournament of order $n$. Let $v_i$ be a vertex with score $s_i$ and imbalance $t_i$. Then $t_i = d_i^+ - d_i^- = s_i - (n - 1 - s_i) = 2s_i - (n - 1)$, or by rearranging $s_i = \frac{n-1+t_i}{2}$. Conversely, assume that $v_i$ is a vertex of a tournament with $n$ vertices and $t_i$ is the imbalance of $v_i$. Let $s_i = \frac{n-1+t_i}{2}$ then $s_i$ is the score of $v_i$. Thus, we have

**Lemma 6.1.1.** Let $t_i$ be the imbalance of a vertex $v_i$ in a tournament. Then $s_i$ is the score of $v_i$ if and only if

$$s_i = \frac{n-1+t_i}{2},$$

where $n$ is the order of the tournament.

A tournament is said to be *regular* if all the vertices have the same score [62]. Clearly, there exists a regular tournament on $n$ vertices with all scores equal to $s$ if and only if $n$ is odd and $s = \frac{n-1}{2}$. Therefore, the imbalance of any vertex $v_i$ of a regular tournament is $t_i = 2s - (n - 1) = 0$ and the imbalance set of any regular tournament is $\{0\}$.

The following is a set of obvious necessary conditions for the imbalance set of a tournament.

**Lemma 6.1.2.** If a finite nonempty set $Z$ of integers is the imbalance set of a tournament of order $n$, then all the elements of $Z$ have the same parity as $n - 1$ and $Z$ either contains at least one positive integer and at least one negative integer, or contains only a single element $0$.

**Proof.** If $Z$ is the imbalance set of a tournament with $n$ vertices, then by Theorem 1.1.3, the elements of $Z$ must have the same parity as $n - 1$. Furthermore, either the tournament is regular and $Z = \{0\}$, or $Z$ must contain at least one positive integer and at least one negative integer so the corresponding imbalance sequence sums to zero. 

The natural question is whether these conditions are also sufficient. The answer is ‘no’ as can be seen from the following example.
Example 6.1.3. Let \( Z = \{6, -10\} \). Then \( Z \) satisfies the necessary conditions given in Lemma 6.1.2 and it can potentially be the imbalance set of a tournament with an odd number of vertices. However, any sequence with elements chosen from \( Z \) can sum to zero only if it consists of an even number of elements (e.g., 6, 6, 6, 6, -10, -10, -10). Thus, by Theorem 1.1.3, we cannot construct a tournament imbalance sequence from \( Z \) and so, \( Z \) is not a tournament imbalance set.

Although the conditions given in Lemma 6.1.2 are not sufficient in general, they are sufficient if \( Z \) consists of odd integers. We first show the following.

Theorem 6.1.4. Let \( X = \{x_1, \ldots, x_l\} \) and \( Y = \{-y_1, \ldots, -y_m\} \) be disjoint nonempty sets, where \( x_1 > \cdots > x_l \) and \( y_1 < \cdots < y_m \) are all positive odd integers. Let \( L = \sum_{i=1}^{l} x_i \), \( M = \sum_{i=1}^{m} y_i \) and \( n = lM + mL \). Then there exists a tournament of order \( n \) with imbalance set \( X \cup Y \).

Proof. We observe that \( n \) is even, and thus all the elements of \( X \cup Y \) have the same parity as \( n - 1 \). Let \( x^{(p)} \) denote the fact that \( x \) appears as \( p \) consecutive terms of a sequence. We use Theorem 1.1.3 to prove that the \( n \)-term sequence

\[
[t_i]_1^n = x_1^{(M)}, \ldots, x_l^{(M)}, -y_1^{(L)}, \ldots, -y_m^{(L)}
\]

is the imbalance sequence of a tournament arranged in nonincreasing order. First, note that

\[
\sum_{i=1}^{M} t_i = Mt_1 = Mx_1 \leq M((l-1)M + mL) = M(n-M),
\]

\[
\sum_{i=1}^{2M} t_i = M(t_1 + t_2) = M(x_1 + x_2) \leq 2M((l-2)M + mL) = 2M(n-2M),
\]

\[
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

\[
\sum_{i=1}^{lM} t_i = M \sum_{i=1}^{l} x_i = LM \leq lM(mL) = lM(n-lM),
\]

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\[
\sum_{i=1}^{lM+L} t_i = LM - Ly_1 \leq (lM + L)(m - 1)L = (lM + L)(n - lM - L),
\]

\[
\sum_{i=1}^{lM+2L} t_i = LM - L(y_1 + y_2) \leq (lM + L)(m - 2)L = (lM + 2L)(n - lM - 2L),
\]

\[\ldots \ldots \ldots \ldots \ldots \]

and

\[
\sum_{i=1}^{n} t_i = M \sum_{i=1}^{l} x_i - L \sum_{i=1}^{m} y_i = 0 = n(n - lM - mL).
\]

Thus, inequality (1.3) holds for \(j = M, 2M, \ldots, lM, lM + L, lM + 2L, \ldots, lM + mL(= n)\) with equality when \(j = n\). Now suppose that for some other value \(j = j_0\) we have \(\sum_{i=1}^{j_0} t_i > j_0(n - j_0)\) and \(j_0\) is the smallest such integer. But then \(t_{j_0} > n - 2j_0 + 1\) and as \(j_0 \neq M, 2M, \ldots, lM, lM + L, lM + 2L, \ldots, n\), we have \(t_{j_0+1} = t_{j_0} > n - 2j_0 + 1 > n - 2j_0 - 1 = n - 2(j_0 + 1) + 1\). Thus, \(\sum_{i=1}^{j_0+1} t_i > (j_0 + 1)(n - j_0 - 1)\), showing that \(j_0 + 1 \neq M, 2M, \ldots, lM, lM + L, lM + 2L, \ldots, n\). Continuing in this way leads to a contradiction as we must reach one of \(M, 2M, \ldots, lM, lM + L, lM + 2L, \ldots, n\) in finitely many steps. \(\square\)

Together Lemma 6.1.2 and Theorem 6.1.4 immediately give the following necessary and sufficient conditions for odd imbalance sets of tournaments.

**Corollary 6.1.5.** A finite nonempty set of odd integers is the imbalance set of a tournament if and only if it contains at least one positive integer and at least one negative integer.

### 6.2 The case of even imbalances

Mubayi, Will and West [138] consider simple digraphs with the maximum number of arcs that realize imbalance sequences. Theorem 6 in [138] proves the following.

**Lemma 6.2.1 ([138]).** Let \(D\) be an oriented graph with the maximum number of arcs realizing the imbalance sequence \([t_i]_1^n\). Then any vertex in \(D\) has at most one non-neighbour and the number of arcs in \(D\) is equal to \(\sum_{i=1}^{n} \left\lceil \frac{n-1+t_i}{2} \right\rceil\).
A partial tournament is an oriented graph obtained by removing one or more arcs from a tournament [60]. We say that a partial tournament of order $n$ is a near tournament if each vertex is joined to all the other vertices except exactly one. Clearly, every near tournament has even order.

In this section, we characterize the sets of even integers that are imbalance sets of tournaments. Recall that $\{0\}$ is the imbalance set of every regular tournament. Therefore, in the remainder of this section, we focus on sets of even integers containing some nonzero element. Example 6.1.3 shows that not every set of even integers that satisfies the necessary conditions of Lemma 6.1.2 is the imbalance set of a tournament. We can nevertheless prove that any such set is the imbalance set of a near tournament.

**Lemma 6.2.2.** Let $X = \{x_1, \ldots, x_l\}$ and $Y = \{-y_1, \ldots, -y_m\}$ be disjoint nonempty sets, where $x_1 > \cdots > x_l$ are non-negative even integers and $y_1 < \cdots < y_m$ are positive even integers. Suppose that $X \cup Y \neq \{0\}$. Let $L = \sum_{i=1}^{l} x_i$, $M = \sum_{i=1}^{m} y_i$ and $n = lM + mL$. Then there exists a near tournament of order $n$ with imbalance set $X \cup Y$.

**Proof.** Since $L$ and $M$ are even, $n$ is even and so we cannot construct a tournament of order $n$ with imbalance set $X \cup Y$. By mirroring the proof of of Theorem 6.1.4, we can show that the $n$-term sequence

$$[t_{i}]_{1}^{n} = x_1^{(M)}, \ldots, x_l^{(M)}, -y_1^{(L)}, \ldots, -y_m^{(L)}$$

is the imbalance sequence of a simple digraph. Let $D$ be an oriented graph with the maximum number of arcs realizing $[t_{i}]_{1}^{n}$. Since all the imbalances $t_i$ are even while $n - 1$ is odd, by Lemma 6.2.1 the number of arcs in $D$ is

$$\sum_{i=1}^{n} \left\lfloor \frac{n - 1 + t_i}{2} \right\rfloor = \sum_{i=1}^{n} \frac{n - 2 + t_i}{2} = \frac{n(n - 2)}{2},$$

which is $\frac{n}{2}$ less than the number of arcs of a tournament of order $n$. Therefore, Lemma 6.2.1
implies that every vertex of $D$ has exactly one non-neighbour, and hence $D$ must be a near tournament.

The following result shows that under certain conditions, we can transform $D$ into a tournament by adding a suitable number of vertices.

**Theorem 6.2.3.** Let $X$, $Y$ and $n$ be as defined in Lemma 6.2.2.

(i) If $0 \in X \cup Y$ then there exists a tournament of order $n + 1$ with imbalance set $X \cup Y$.

(ii) If there exists an $x_p \in X$ and (not necessarily distinct) $-y_q, -y_r \in Y$ such that $x_p = y_q + y_r$ then there exists a tournament of order $n + 3$ with imbalance set $X \cup Y$.

(iii) If there exists a $-y_p \in Y$ and (not necessarily distinct) $x_q, x_r \in X$ such that $y_p = x_q + x_r$ then there exists a tournament of order $n + 3$ with imbalance set $X \cup Y$.

**Proof.** Let $T$ be a near tournament realizing the imbalance sequence $[t_i]_n$ as described in the proof of Lemma 6.2.2. For a vertex $v_i$ in $T$ let $v'_i$ denote the unique non-neighbour of $v_i$. Also, let $(u, v)$ denote an arc directed from vertex $u$ to vertex $v$. In the three cases, we can transform $T$ into a tournament as follows.

(i) Add a vertex $v$ to $T$ in such a way that for every pair of non-adjacent vertices $v_i$ and $v'_i$ we insert the arcs $(v_i, v'_i)$, $(v'_i, v)$ and $(v, v_i)$. Thus, the imbalance of all the vertices of $T$ is preserved and the new vertex $v$ has imbalance 0. Since every vertex of $T$ has been linked with every other vertex of $T$ as well as the new vertex $v$, the resulting digraph is a tournament.

(ii) Add three new vertices $u_1$, $u_2$ and $u_3$ to $T$ and insert arcs in the following manner:

1. Insert $(u_1, u_2)$, $(u_2, u_3)$ and $(u_3, u_1)$.

2. Choose any $\frac{x_p}{2} = \frac{y_q + y_r}{2}$ pairs $\{v_i, v'_i\}$ of non-neighbouring vertices in $T$ and insert $(v_i, v'_i)$. Out of these, choose any $\frac{y_p}{2}$ pairs. For each of these pairs insert the arcs $(u_1, v_i)$, $(u_1, v'_i)$, $(v_i, u_2)$, $(v'_i, u_2)$, $(v'_i, u_3)$ and $(u_3, v_i)$. For the other $\frac{y_p}{2}$ pairs insert the arcs $(u_1, v_i)$, $(u_1, v'_i)$, $(v_i, u_3)$, $(v'_i, u_3)$, $(v'_i, u_2)$ and $(u_2, v_i)$.
3. For the remaining \( \frac{n-x_p}{2} \) pairs \( \{v_i, v'_i\} \) of non-neighbours, insert the arcs \((u_1, v_i), (v'_i, u_1), (v_i, v'_i), (v_i, u_2), (u_2, v'_i), (u_3, v_i) \) and \((v'_i, u_3)\).

Since every vertex is joined with every other vertex, the resulting digraph is a tournament. Furthermore, the imbalance of each vertex of \( T \) is preserved, while the new vertices \( u_1, u_2 \) and \( u_3 \) have imbalances \( x_p, -y_q \) and \( -y_r \) respectively.

\((\text{iii})\) The proof is essentially the same as that of case (ii). \(\square\)

Theorem 6.2.3 can be generalized, and it is the generalized version that is of interest to us as it leads to the characterization of even imbalance sets of tournaments. However, we stated and proved Theorem 6.2.3 to provide the reader with a concrete perspective of what is happening in the more complex setting of Theorem 6.2.4.

**Theorem 6.2.4.** Let \( X, Y \) and \( n \) be as defined in Lemma 6.2.2. The set \( X \cup Y \) is the imbalance set of a tournament if any one of the following conditions is satisfied:

\((i)\) \( 0 \in X \cup Y \).
(ii) There exist an odd number of (not necessarily distinct) \( x_{p_1}, \ldots, x_{p_{2r+1}} \in X \) and an even number of (not necessarily distinct) \(-y_{q_1}, \ldots, -y_{q_{2s}} \in Y\) such that \( \sum_{j=1}^{2r+1} x_{p_j} = \sum_{j=1}^{2s} y_{q_j} \).

(iii) There exist an odd number of (not necessarily distinct) \(-y_{p_1}, \ldots, -y_{p_{2r+1}} \in Y\) and an even number of (not necessarily distinct) \( x_{q_1}, \ldots, x_{q_{2s}} \in X\) such that \( \sum_{j=1}^{2r+1} y_{p_j} = \sum_{j=1}^{2s} x_{q_j} \).

Proof. Let \( T \) be a near tournament realizing the imbalance sequence \([t_i]_1^n\) as defined in the proof of Lemma 6.2.2.

(i) The proof is the same as part (i) of Theorem 6.2.3.

(ii) Add \( 2r + 2s + 1 \) new vertices labelled \( u_{p_1}, \ldots, u_{p_{2r+1}}, u_{q_1}, \ldots, u_{q_{2s}} \) to \( T \). Note that in the construction that follows we will relabel them in different ways, such as \( u_1, \ldots, u_{2r+2s+1} \), when convenient. We insert arcs using the following procedure.

**ADD ARCS:**

1. Insert arcs so that the newly added vertices induce a regular tournament of order \( 2r + 2s + 1 \).
2. Choose any \( \frac{\sum_{j=1}^{2r+1} x_{p_j}}{2} = \frac{\sum_{j=1}^{2s} y_{q_j}}{2} \) pairs \( \{v_i, v'_i\} \) of non-neighbouring vertices in \( T \) and order them arbitrarily. For each \( i = 1, \ldots, \frac{\sum_{j=1}^{2r+1} x_{p_j}}{2} \) insert the arc \( (v_i, v'_i) \). Then the imbalances of \( v_i \) and \( v'_i \) change by +1 and −1 respectively, for all \( i \).

3. For each \( j = 1, \ldots, 2r+1 \), choose \( i = \frac{\sum_{h=1}^{q_j-1} x_{p_h}}{2} + 1, \ldots, \frac{\sum_{h=1}^{q_j} y_{q_j}}{2} \), where \( p_{j_0} = 1 \), and insert \( x_{p_j} \) arcs \( (u_{p_j}, v_i) \) and \( (u_{p_j}, v'_i) \). This gives \( u_{p_j} \), the imbalance \( x_{p_j} \).

4. For each \( j = 1, \ldots, 2s \) choose \( i = \frac{\sum_{h=1}^{y_{q_j}} y_{h}}{2} + 1, \ldots, \frac{\sum_{h=1}^{y_{q_j}} y_{q_j}}{2} \), where \( q_{j_0} = 1 \), and insert the arcs \( (v_i, u_{q_j}) \), \( (v'_i, u_{q_j}) \). Then the imbalance of \( u_{q_j} \) is \(-y_{q_j}\), but the imbalances of \( v_i \) and \( v'_i \) are still perturbed by +1 and −1 respectively, for \( i = 1, \ldots, \frac{\sum_{j=1}^{2r+1} x_{p_j}}{2} \).

5. For every \( i = 1, \ldots, \frac{\sum_{j=1}^{2r+1} x_{p_j}}{2} \), list the \( u_j \)'s that are not already linked with \( v_i \) and \( v'_i \). There are exactly \( 2r + 2s - 1 \) such \( u_j \)'s, for each \( i \). Label them arbitrarily from 1, \( \ldots, 2r + 2s - 1 \). For \( j = 1, \ldots, \frac{2r+2s+1}{2} \) insert the arcs \( (u_j, v_i) \) and \( (u'_j, u_j) \). For \( j = \frac{2r+2s+1}{2} + 1, \ldots, 2r + 2s - 2 \) insert the arcs \( (u_j, v'_i) \) and \( (v_i, u_j) \). Finally, insert the arcs \( (u_{2r+2s-1}, v_i) \) and \( (v'_i, u_{2r+2s-1}) \). This preserves all the imbalances.

6. For the remaining \( \frac{n-\sum_{j=1}^{2r+1} x_{p_j}}{2} \) pairs \( \{v_i, v'_i\} \) of non-neighbours, insert the arc \( (v_i, v'_i) \). Label all the \( u \)'s arbitrarily from 1, \( \ldots, 2r + 2s + 1 \). For \( j = 1, \ldots, \frac{2r+2s}{2} \), insert the arcs \( (u_j, v_i) \) and \( (v'_i, u_j) \). For \( j = \frac{2r+2s}{2} + 1, \ldots, 2r + 2s \), insert the arcs \( (u_j, v_i) \) and \( (v'_i, u_j) \). Finally, insert the arcs \( (u_{2r+2s+1}, v_i) \) and \( (v'_i, u_{2r+2s+1}) \). This preserves all the imbalances.

Since every vertex is joined with every other vertex, the resulting digraph is a tournament. Furthermore, the imbalance of each vertex of \( T \) is preserved, while the new vertices \( u_{p_1}, \ldots, u_{p_{2r+1}}, u_{q_1}, \ldots, u_{q_{2s}} \) have imbalances \( x_{p_1}, \ldots, x_{p_{2r+1}}, -y_{q_1}, \ldots, -y_{q_{2s}} \), respectively.

(iii) The proof is essentially the same as that of case (ii).

The reader can easily draw parallels between the proofs of Theorems 6.2.3 (ii) and 6.2.4 (ii). For instance, the first and the last steps of both proofs are essentially achieving the
same target while steps 2-5 of the latter are similar to, but more sophisticated than step 2 of the former.

Analyzing the above proof leads to two simpler sufficient conditions for tournament imbalance sets. The first one is a fairly straightforward consequence of Theorem 6.2.3 (i).

**Corollary 6.2.5.** If $Z$ is the empty set or contains at least one positive integer and at least one negative even integer, then $Z \cup \{0\}$ is the imbalance set of a tournament.

The second condition is not as obvious and is more of an arithmetic result than a combinatorial one. First, we note that for any positive integer $p \geq 1$, the set $\{2^p, -2^p\}$ is not a tournament imbalance set as any zero-sum sequence formed by the elements of this set necessarily consists of an even number of elements. However, the following sufficient condition shows that any other set of positive and negative even integers containing a power of 2 is a tournament imbalance set.

**Corollary 6.2.6.** Let $Z$ be a finite nonempty set of even integers containing at least one positive integer and at least one negative integer. Suppose $Z$ contains an element of the form $2^p$ or $-2^p$, for some positive integer $p \geq 1$, and $Z \neq \{2^p, -2^p\}$. Then $Z$ is the imbalance set of a tournament.
Proof. Let us assume that for some positive integer \( p \geq 1 \), \( 2^p \) is an element of \( Z \). Choose any negative element \( -y \in Z \). Then \( y = r2^q \), where \( q \geq 1 \) is a positive integer and \( r \geq 1 \) is an odd positive integer such that if \( r = 1 \) then \( q \neq p \). (If this is not possible, we can start with \( -2^p \in Z \) and choose an \( x = r2^q \) from \( Z \) with \( q \neq p \).) If \( \max\{p,q\} \), we have

\[
\underbrace{2^p + \cdots + 2^p}_{r \text{ terms}} = \underbrace{y + \cdots + y}_{2^p - \text{ terms}},
\]

and by Theorem 6.2.4, \( Z \) is a tournament imbalance set. If \( q = \max\{p,q\} \), we have

\[
\underbrace{2^p + \cdots + 2^p}_{r2^p - \text{ terms}} = y,
\]

and again by Theorem 6.2.4, \( Z \) is a tournament imbalance set.

After deriving several sufficient conditions for tournament imbalance sets of even integers, the natural question is whether the sufficient conditions given in Theorem 6.2.4 are also necessary. The answer is positive as seen from the following result.

**Corollary 6.2.7.** Let \( Z = X \cup Y \) be a finite nonempty set of even integers, where \( X \) is a set of non-negative integers and \( Y \) is a set of negative integers. Then \( Z \) is the imbalance set of a tournament if and only if either \( Z = \{0\} \), or both \( X \) and \( Y \) are nonempty and satisfy one of the conditions (i), (ii) or (iii) of Theorem 6.2.4.

**Proof.** The sufficiency follows from Theorem 6.2.4.

To prove necessity, suppose that \( 0 \notin X \cup Y \) and let \( X \cup Y \) be the imbalance set of a tournament of order \( k \). This implies that we can form a sequence \([t_i]_1^k\) consisting of an odd number of not necessarily distinct terms from the elements of \( X \cup Y \) that sums to zero. Since \( k \) is odd, either the number of terms from \( X \) is odd or the number of terms from \( Y \) is odd, but not both. Thus we have an odd (respectively, even) number of terms \( x \in X \) and an even (respectively odd) number of terms \( -y \in Y \) such that \( \sum x = \sum y \). This shows that either part (ii) or part (iii) of Theorem 6.2.4 is satisfied.
6.3 Equal-sum subsets and sequences

In this section, we introduce a new problem namely, EQUAL-SUM SEQUENCES (ESSeq) as a generalization of EQUAL-SUM SUBSETS (ESS). The ideas presented here play a key role in Section 6.4, where we develop an algorithm to solve TOURNAMENT IMBALANCE SET (TIS).

Given a set of non-negative integers, we call the search problem of finding two disjoint nonempty subsets that have identical sums EQUAL-SUM SUBSETS (ESS). Several authors [11, 167] have considered this, and the corresponding decision and optimization problems. Additionally, many variants of ESS have been studied in literature. For instance, if we require subsets to be found from two different sets of non-negative integers, the problem is called EQUAL-SUM SUBSETS FROM TWO SETS (ESST) [64, 63]. It is known that ESS is NP-hard [167] and so is ESST [64]. Here we are interested in the following variation of ESS and ESST.

**Definition 6.3.1 (EQUAL-SUM SEQUENCES (ESSeq)).** Given two finite nonempty sets X and Y of non-negative integers, and a positive integer k, find two nonempty finite sequences [x] and [y] consisting of elements from X and Y respectively, with each element allowed to repeat at the most k times, such that \( \sum x = \sum y \).

ESSeq is a natural generalization of ESS and ESST. Note that if \( k \geq \max\{\min X, \min Y\} \), we can trivially construct the sequence [x] of length \( \min Y \) from X by just repeating the term \( x = \min X \), and the sequence [y] of length \( \min X \) from Y by repeating the term \( y = \min Y \), such that the sums of the two sequences are the same. Thus, we have the following observation.

**Proposition 6.3.2.** For any finite nonempty sets X and Y of non-negative integers, if \( k \geq \max\{\min X, \min Y\} \) then ESSeq\((X, Y, k)\) is a ‘Yes’ instance of the ESSeq decision problem.

We now study the complexity of ESSeq in general. Clearly, ESSeq is in class NP. Furthermore, ESST corresponds to the special case \( k = 1 \) of ESSeq. Therefore, we have the following.
Proposition 6.3.3. The ESSeq search (decision) problem is NP-hard (NP-complete).

Consider an instance ESSeq\((X, Y, k)\) of ESSeq and let \(X^{(k)}\) and \(Y^{(k)}\) denote the multisets consisting of elements of \(X\) and \(Y\), respectively, such that each element is repeated \(k\) times. Then one can solve ESSeq\((X, Y, k)\) by running through all possible submultisets of \(X^{(k)}\) and \(Y^{(k)}\) in \(O(2^{k\max\{|X|,|Y|\}})\) time. Let us call the resulting algorithm, which finds all possible solutions to an ESSeq instance, EQUAL SEQ.

6.4 Algorithmic aspects

The aim of this section is to study the Tournament Imbalance Set (TIS) problem (see Definition 1.1.4) and present an algorithm that generates a tournament realizing any tournament imbalance set. We achieve this by observing the connection between TIS and ESSeq that was introduced and discussed in Section 6.3, showing how the algorithm EQUAL SEQ discussed at the end of Section 6.3 can be used to solve even instances of TIS. First, we prove a result that gives an upper bound on the minimum length of equal-sum sequences (if they exist) obtained from two finite non-empty sets of non-negative integers.

Theorem 6.4.1. Let \(X, Y, l, m, L, M\) and \(n\) be as defined in Lemma 6.2.2. If \(k = p + q\) is the least odd number such that there exists a \(p\)-term sequence from \(X\) and a \(q\)-term sequence from \(-Y = \{y : -y \in Y\}\) having the same sum, then \(k < n\).

Proof. We observe that \(k\) equals the minimal length of a zero-sum sequence of odd length from \(X \cup Y\). We prove the result by induction on \(n\). Note that according to the conditions of Lemma 6.2.2, \(X \cup Y \neq \{0\}\) and so the minimum possible value of \(n\) is 4, which corresponds only to the sets \(X = \{2\}\) and \(Y = \{-2\}\). Since it is not possible to form a zero-sum sequence with an odd number of terms from \(\{2, -2\}\), we disregard this case. The next smallest value of \(n\) is 6, which corresponds to \(X \cup Y = \{2, 0, -2\}, \{2, -4\}, \{4, -2\}\). Each of these sets admits a zero-sum sequence of length \(k = 3\) and so \(k < n\).
Now we aim to show that the result holds for any $n > 6$ by assuming that it holds for all values less than $n$. Let $X$ and $Y$ be any two sets of integers corresponding to $n$ and let $k \geq 3$ be the minimum odd number such that there exists a $k$-term zero-sum sequence $a_1, \ldots, a_p, -b_1, \ldots, -b_q$, where $k = p + q$, $a_1 > \cdots > a_p \in X$ and $-b_1 > \cdots > -b_q \in Y$. Assume that $k \geq n$. Then $k > 6$ and therefore, either $p \geq 4$ or $q \geq 4$.

In the former case, the sequence $a_1 + a_2, a_3 + a_4, -b_1, \ldots, -b_q$ is a zero-sum sequence of odd length $k' = k - 2$ corresponding to the sets $X' = (X \setminus \{a_1, a_2, a_3, a_4\}) \cup \{a_1 + a_2, a_3 + a_4\}$ and $Y' = Y$. Let $n'$ be the value of the parameter $n$ corresponding to the sets $X'$ and $Y'$. Then we have $n' \leq n - 2$ and so $k' = k - 2 \geq n - 2 \geq n'$, contradicting the induction hypothesis.

In the latter case, the sequence $a_1, \ldots, a_p, -b_1, \ldots, -b_q$ is a zero-sum sequence of odd length $k' = k - 2$ corresponding to the sets $X' = X$ and $Y' = (Y \setminus \{-b_q, -b_{q-2}, -b_{q-1}, -b_q\}) \cup \{-b_{q-3}, -b_{q-2}, -b_{q-1}, -b_q\}$. For these sets we have $n' \leq n - 2$ and so $k' \geq n'$, again contradicting the induction hypothesis.

We now present an algorithm that solves both the decision and search versions of TIS. Our algorithm is based on the proofs of Theorems 6.1.4, 6.2.4 and 6.4.1. First, we form a suitable $n$-term imbalance sequence $[t_i]^n$ of a digraph that is not necessarily a tournament. Lemma 6.2.1 is then used to construct an oriented graph, with the maximum possible number of arcs, realizing this imbalance sequence. The idea is to start with an arbitrary vertex $v$ having imbalance $t_i$ and attach it to $\lceil \frac{n-1+t_i}{2} \rceil$ vertices by arcs directed away from $v$. If $t_i$ has the same parity as $n - 1$ then $v$ is joined with $n - 1 - \lceil \frac{n-1+t_i}{2} \rceil$ other vertices by arcs directed towards $v$. Otherwise, it is joined with $n - 2 - \lceil \frac{n-1+t_i}{2} \rceil$ other vertices by arcs directed towards $v$. Thus, $v$ is joined to every vertex except possibly one. These steps are then repeated for every vertex without attaching any new arcs to the preprocessed vertices. We name this $O(n^2)$ procedure Max Realization [138].

Now suppose that $Z$ is a finite nonempty set of integers. Form the sets $X = \{z \in Z : z \geq 0\} = \{x_1, \ldots, x_l\}$ and $Y = \{z \in Z : z < 0\} = \{-y_1, \ldots, -y_m\}$ arranged in decreasing order.
Let \( L = \sum_{i=1}^{l} x_i \), \( M = \sum_{i=1}^{m} y_i \) and \( n = lM + mL \) as in the earlier proofs. The following algorithm outputs a tournament that realizes \( Z \), whenever such a tournament exists. It makes use of the procedures ADD ARCS, EQUAL SEQ and MAX REALIZATION discussed earlier.

**Algorithm 6.4.2 (Imbalance Set).**

1. If either \( X \) or \( Y \) is empty, then \( Z \) is not a tournament imbalance set. Stop.

2. If elements of \( Z \) have different parity, \( Z \) is not a tournament imbalance set. Stop.

3. Form the sequence \([t_i]_n = x_1^{(M)}, \ldots, x_l^{(M)}, -y_1^{(L)}, \ldots, -y_m^{(L)}\).

4. Call the procedure MAX REALIZATION to realize \([t_i]_n\) as an oriented graph \( D \) with the maximum number of arcs.

5. If elements of \( Z \) have odd parity, output \( D \). End.

6. If elements of \( Z \) have even parity, call EQUAL SEQ with the input \((X, -Y, n - 1)\) to find sequences \([x]_a\) and \([y]_b\), with \( a \) and \( b \) having different parity and \( \sum x = \sum y \). If no such sequences exist then \( Z \) is not a tournament imbalance set. End.

7. Add \( 2a + 2b + 1 \) isolated vertices to \( D \).

8. Call ADD ARCS to add arcs to \( D \) to form a tournament \( T \). Return \( T \).

**Proposition 6.4.3.** Algorithm 6.4.2 is correct and runs in \( O(2^{n \max\{l,m\}}) \) time.

**Proof.** The correctness follows immediately from Theorem 6.1.4, Lemma 6.2.2, Theorem 6.2.4 and Theorem 6.4.1. In particular, Theorem 6.4.1 guarantees that when \( Z \) is a set of even integers, step 6 of Algorithm 6.4.2 necessarily finds the required sequences if they exist. Now note that the computational complexity of Algorithm 6.4.2 is dominated by steps 4 and 6. Step 4 can be performed in \( O(n^2) = O((lM + mL)^2) \) (pseudo-polynomial) time, whereas step 6 takes \( O(2^{k \max\{|X|,|Y|\}}) = O(2^{n \max\{l,m\}}) \) time. The overall complexity is \( O(2^{n \max\{l,m\}}) \). \( \square \)
Thus, we can use Algorithm 6.4.2 to check if a given set of integers is the imbalance set of a tournament and, if so, to construct a tournament realizing the set. We now illustrate Algorithm 6.4.2 by showing how it generates a tournament realizing the imbalance set \( \{4, 2, -2\} \).

**Example 6.4.4.** Consider the set \( Z = \{4, 2, -2\} \). Since \( Z \) satisfies the conditions in the first two steps of Algorithm 6.4.2, the algorithm goes to step 3 and forms the sequence 4, 4, 2, 2, -2, -2, -2, -2, -2, -2. Step 4 calls the procedure \texttt{Max Realization} to output an oriented graph of order 10 realizing \( Z \). However, this oriented graph is only a near tournament and not a tournament (see the digraph induced by the black vertices in Figure 6.4). Since the elements of \( Z \) have even parity, the algorithm proceeds to step 6 and finds the sequences \( [x_i]_1 = 4 \) and \( [y_i]^2_1 = -2, -2 \), with odd and even number of terms respectively, such that \( \sum x_i = -\sum y_i \). Then step 7 adds 3 new vertices (colored white in Figure 6.4) to
the near tournament obtained in step 4. In the end, step 8 adds the arcs (dashed arcs in Figure 6.4) necessary to form a tournament of order 13 in such a way that the imbalances of the old vertices are preserved and the new vertices have imbalances \(4, -2, \) and \(-2\). The output of Algorithm 1 is a tournament with imbalance sequence \(4, 4, 4, 2, 2, -2, -2, -2, -2, -2, -2, -2, -2\) as shown in Figure 6.4.

The results presented in the earlier sections can be used to estimate the order of a tournament realizing an imbalance set. Let us denote by \(\text{ord}(Z)\) the minimal order of a tournament realizing an imbalance set \(Z\).

**Corollary 6.4.5.** Let \(Z\) be a tournament imbalance set (i.e., it satisfies the conditions of Corollary 6.1.5 or Corollary 6.2.7). Define \(X, Y, l, m, L, M\) and \(n\) as in the earlier results.

(i) If \(Z\) consists of odd integers then \(\text{ord}(Z) \leq n = lM + mL\).

(ii) If \(Z\) consists of even integers and \(0 \in Z\) then \(\text{ord}(Z) \leq n + 1 = lM + mL + 1\).

(iii) If \(Z\) consists of even integers and \(0 \notin Z\) then \(\text{ord}(Z) < 2n = 2(lM + mL)\).

**Proof.** The proof of (i) follows from Theorem 6.1.4, while (ii) follows from Theorem 6.2.3 (i). For (iii), observe that the order of the tournament constructed in the proof of Theorem 6.2.4 is \(n + 2r + 2s + 1\). From Theorem 6.4.1, \(2r + 2s + 1 < n\). As a result, the constructed tournament has order at most \(2n - 1\).
### Chapter 7

**Regularity and random generation of hypertournaments**

In this chapter, we study hypertournaments. First, we present some characterizations of regularity in hypertournaments and then address an important algorithmic question.

The “degree sequence problem” is one of the most fundamental questions in algorithmic graph theory. It asks to generate all simple graphs realizing a degree sequence uniformly at random. Over the years, a number of Markov chain Monte Carlo (MCMC) algorithms have been proposed [10, 67, 104, 105, 134] that achieve this for different classes of graphs and hypergraphs. The biggest challenge is to guarantee ‘rapid mixing’ of the Markov chain, which means the chain reaches uniform stationary distribution in time bounded by a polynomial in the size of the input. Here we build a rapidly mixing Markov chain for uniformly sampling hypertournaments with a given score sequence.

We investigate the problem of randomly sampling all $k$-hypertournaments with a given score (equivalently losing score) sequence. The incidence matrix of a $k$-hypertournament is called a $k$-hypertournament matrix. We construct a Markov chain, denoted by $\mathcal{M}_{mc}$ on the set $\mathcal{M}_S$ of all $k$-hypertournament matrices with the same score sequence $S$ (see Definition 1.1.8) based on a switching operation. We prove that $\mathcal{M}_{mc}$ is ergodic, has a uniform stationary
distribution and is rapidly mixing. We then use $M_{mc}$ to construct a Markov chain $H_{mc}$ on the set $H_S$ of all $k$-hypertournaments with score sequence $S$ and prove ergodicity, uniform stationary distribution and rapid mixing of the new Markov chain. Some results from this chapter have appeared in [105, 106].

7.1 Regularity in hypertournaments

A hypertournament is said to be *regular* if all vertices have the same score (equivalently the same losing score) [106, 113]. Koh and Ree [113] prove the following necessary and sufficient condition for the existence of regular $k$-hypertournaments on $n$ vertices.

**Theorem 7.1.1.** For $n \geq 3$ and $2 \leq k \leq n - 1$, a regular $k$-hypertournament on $n$ vertices exists if and only if $n$ divides $\binom{n}{k}$.

In this section, we present a new proof of Theorem 7.1.1. In addition, we present three results that characterize regularity of a $k$-hypertournament in different ways. The first of these results gives a lower bound on $\sum_{i=1}^{j} r_i^g$, where $1 < g < \infty$ is a real number and $[r_i]_1^n$ is the losing score sequence of a $k$-hypertournament.

**Theorem 7.1.2.** Let $n$ and $k$ be two non-negative integers with $n \geq k > 1$. If $R = [r_i]_1^n$ is the losing score sequence of a $k$-hypertournament, then for $1 < g < \infty$

$$\sum_{i=1}^{j} r_i^g \geq \frac{j}{k^g} \left( \frac{j-1}{k-1} \right)^g,$$

where $1 \leq j < n$. Moreover,

$$\sum_{i=1}^{n} r_i^g \geq \frac{n}{k^g} \left( \frac{n-1}{k-1} \right)^g,$$

(7.1)

with equality in (7.1) if and only if the hypertournament is regular.
Proof. By Theorem 1.1.6, we have

\[ \binom{j}{k} \leq \sum_{i=1}^{j} r_i, \]

for 1 ≤ j < n. By Hölder’s inequality,

\[ \binom{j}{k} \leq \sum_{i=1}^{j} r_i = \sum_{i=1}^{j} r_i \cdot 1 \leq \left( \sum_{i=1}^{j} r_i^g \right)^{\frac{1}{g}} \left( \sum_{i=1}^{j} 1^h \right)^{\frac{1}{h}}, \]

for 1 ≤ j < n, where \( \frac{1}{g} + \frac{1}{h} = 1 \).

Thus,

\[ \binom{j}{k} \leq \left( \sum_{i=1}^{j} r_i^g \right)^{\frac{1}{g}} \left( \sum_{i=1}^{j} 1^h \right)^{\frac{1}{h}} = \left( \sum_{i=1}^{j} r_i^g \right)^{\frac{1}{g}} j^{\frac{1}{h}}, \]

that is,

\[ j^{\frac{1}{h}} \left( \frac{j!}{k!(j-k)!} \right) \leq \left( \sum_{i=1}^{j} r_i^g \right)^{\frac{1}{g}}, \]

or

\[ \frac{j^{1-\frac{1}{h}}}{k} \left( \frac{(j-1)!}{(k-1)!(j-k)!} \right) \leq \left( \sum_{i=1}^{j} r_i^g \right)^{\frac{1}{g}}. \]

Hence,

\[ \sum_{i=1}^{j} r_i^g \geq \frac{j}{k^g} \left( \frac{j-1}{k-1} \right)^g. \tag{7.2} \]

For j = n, Theorem 1.1.6 gives

\[ \binom{n}{k} = \sum_{i=1}^{n} r_i, \]

and inequalities (7.2) become

\[ \sum_{i=1}^{n} r_i^g \geq \frac{n}{k^g} \left( \frac{n-1}{k-1} \right)^g, \]

with equality if and only if \( r_1 = r_2 = \cdots = r_n \) (the condition of equality in Hölder’s inequality), that is, if and only if the hypertournament is regular. \qed

The second result combines scores and losing scores and gives an upper bound for the
usual inner product of a score vector and losing score vector of a hypertournament in $\mathbb{R}^n$. Note that the usual inner product of two vectors $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ is denote by $\langle A, B \rangle$ and defined as $\langle A, B \rangle = a_1b_1 + \cdots + a_nb_n$. It should be noted that Theorem 7.1.3 does not depend on the order of the sequences $[s_i]$ and $[r_i]$ and holds for an arbitrary ordering of scores and losing scores.

**Theorem 7.1.3.** Let $n$ and $k$ be two non-negative integers with $n \geq k > 1$. If $S = [s_i]^n_1$ and $R = [r_i]^n_1$ are respectively the score sequence and losing score sequence of a $k$-hypertournament, both arranged according to the same arbitrary vertex labelling, then

$$\langle S, R \rangle \leq \frac{k-1}{k} \binom{n}{k} \binom{n-1}{k-1},$$

with equality if and only if the hypertournament is regular.

**Proof.** By the linearity of the inner product, $\langle S + R, R \rangle = \langle S, R \rangle + \langle R, R \rangle$. Since $s_i + r_i = \binom{n-1}{k-1}$, this gives

$$\binom{n-1}{k-1} \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_ir_i + \sum_{i=1}^{n} r_i^2.$$ 

From Theorem 1.1.6,

$$\binom{n-1}{k-1} \binom{n}{k} = \sum_{i=1}^{n} s_ir_i + \sum_{i=1}^{n} r_i^2.$$ 

Since the arithmetic mean of $n$ non-negative real numbers never exceeds their root mean square, we obtain

$$\sqrt{\frac{\sum_{i=1}^{n} r_i^2}{n}} \geq \frac{\sum_{i=1}^{n} r_i}{n},$$

with equality if and only if $r_1 = r_2 = \cdots = r_n$, or

$$\sum_{i=1}^{n} r_i^2 \geq \left( \frac{\sum_{i=1}^{n} r_i}{n} \right)^2.$$
Thus,
\[
\binom{n-1}{k-1} \binom{n}{k} \geq \sum_{i=1}^{n} s_i r_i + \frac{\left( \sum_{i=1}^{n} r_i \right)^2}{n} = \sum_{i=1}^{n} s_i r_i + \frac{n}{n} = \sum_{i=1}^{n} s_i r_i + \frac{n}{k} = \frac{k-1}{k} \binom{n}{k} \binom{n-1}{k-1}.
\]
(by Theorem 1.1.6)

Rearranging yields
\[
\sum_{i=1}^{n} s_i r_i \leq \binom{n-1}{k-1} \binom{n}{k} - \frac{\left( \sum_{i=1}^{n} r_i \right)^2}{n} = \frac{k-1}{k} \binom{n}{k} \binom{n-1}{k-1}.
\]

Equality holds if and only if \( r_1 = r_2 = \cdots = r_n \), that is, if and only if the hypertournament is regular. \( \Box \)

An \((x, y)\)-path of length \( t \) [170] in a \( k \)-hypertournament \( H \) is a sequence
\[
(x =) v_1 e_1 v_2 e_2 v_3 \cdots v_{t-1} e_{t-1} v_t (= y)
\]

of distinct vertices \( v_1, v_2, \ldots, v_t, \ t \geq 1 \), and distinct arcs \( e_1, e_2, \ldots, e_{t-1} \) such that for \( 1 \leq i \leq t-1 \), vertex \( v_i \) lies on \( e_i \) in one of the first \( k-1 \) positions and vertex \( v_{i+1} \) occurs as the last element in \( e_i \). Often we express such a path as a list of vertices \( v_1 v_2 v_3 \cdots v_{t-1} v_t \). Our third result provides some insight into the structure of hypertournaments. More precisely, it gives an upper bound on the number of directed paths of length 2 in a \( k \)-hypertournament. Once again regularity is characterized as the condition of equality in the upper bound. This generalizes the well-known upper bound of \( \frac{n(n-1)^2}{4} \) on the number of directed paths of length 2 in a tournament of order \( n \).

**Corollary 7.1.4.** The number of directed paths of length two in a \( k \)-hypertournament never exceeds \( \frac{k-1}{k} \binom{n}{k} \binom{n-1}{k-1} \). The bound is achieved if and only if the hypertournament is regular.

**Proof.** For \( 1 \leq i \leq n \), the quantity \( r_i s_i \) counts the number of distinct directed paths of length two of the form \( xv_i y \). Thus, \( \sum_{i=1}^{n} r_i s_i \) is the total number of distinct directed paths of length two in the hypertournament. By (7.3) this quantity is bounded above by \( \frac{k-1}{k} \binom{n}{k} \binom{n-1}{k-1} \) with equality if and only if the hypertournament is regular. \( \Box \)
Finally, we give a short proof of Theorem 7.1.1 based on Theorem 7.1.2.

Proof of Theorem 7.1.1. If there exists a regular $k$-hypertournament with losing score sequence $[r_i]_1^n$, then by the case of equality for $g = 2$ in (7.2),

$$\sum_{i=1}^{n} r_i^2 = \frac{n}{k^2} \left( \frac{n-1}{k-1} \right)^2,$$

where $r_1 = r_2 = \cdots = r_n = r$. Thus $r^2 = \frac{1}{k^2} \left( \frac{n-1}{k-1} \right)^2$ and so $k$ divides $\binom{n-1}{k-1}$. Now $\frac{n}{k} = \frac{\binom{n-1}{k-1}}{k}$, which implies that $n$ divides $\binom{n}{k}$.

Conversely, assume that $n$ divides $\binom{n}{k}$. Then $k$ divides $\binom{n-1}{k-1}$. For $1 \leq i \leq n$, set $r_i = \frac{1}{k} \binom{n-1}{k-1}$. Then for $1 \leq j \leq n$,

$$\sum_{i=1}^{j} r_i = \frac{j}{k} \binom{n-1}{k-1} = \frac{j(n-1)!}{(n-k)!k!k!} \geq \frac{j!}{(j-k)!k!},$$

with equality when $j = n$. Thus, by Theorem 1.1.6, $[r_i = r_1^n]$ is the losing score sequence of a regular $k$-hypertournament.

\[\square\]
7.2 Switchable configurations in \( k \)-hypertournament matrices

Let \( M(H) \) be a \( k \)-hypertournament matrix. A switchable configuration is one of the four configurations given in Figure 7.1, or a configuration obtained by permuting the rows or columns of one of these four configurations. The operation of switching consists of interchanging the columns of a switchable configuration.

**Example 7.2.1.** Consider the 3-hypertournament \( H_3 = (V, E) \) with \( V = \{v_1, v_2, v_3, v_4\} \) and \( E = \{e_1, e_2, e_3, e_4\} \), where \( e_1 = (v_1, v_2, v_3) \), \( e_2 = (v_1, v_2, v_4) \), \( e_3 = (v_1, v_4, v_3) \) and \( e_4 = (v_2, v_4, v_3) \). The 3-hypertournament matrix associated with \( H_3 \) is given by

\[
M(H_3) = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
-1 & 0 & -1 & -1 \\
0 & -1 & 1 & 1
\end{bmatrix}
\]

Figure 7.2 shows the result of applying some switching operations to \( M(H_3) \).

Evidently, every switching preserves the row and column sums of a \( k \)-hypertournament matrix. Hence, switching preserves the score sequence and the set \( \mathcal{M}_S \) is closed under the switching operations. In addition, every switching is completely reversible, since any switching of a \( k \)-hypertournament matrix \( M(H) \) of type \( S1 \) can be reversed by applying another switching of type \( S1 \) to the resulting matrix and likewise for type \( S2, S3 \) and \( S4 \).

**Lemma 7.2.2.** Every switching is an inversion and hence, is reversible.

It should be noted that all switchable configurations in a \( k \)-hypertournament matrix \( M(H) \) can be identified in \( O(m^2n^3) \) worst case time, where \( m \) is the number of arcs of the hypertournament, by the following algorithm.
Figure 7.2: Some switchings of the 3-hypertournament matrix $M(H_3)$. The switched entries have been written in bold font. Note that $M(H_3)$ has no switchable configuration of type $S3$.

Algorithm 7.2.3.

1. Start with any entry, say $m_{ij}$ of the matrix $M(H)$ and identify all columns $q$ such that $m_{ij} \neq m_{iq}$. Let $Q = \{q : m_{ij} \neq m_{iq}\}$. This step takes $O(m)$ time.

2. For every $q \in Q$, identify all rows $p$ such that $m_{ij} + m_{pj} = m_{iq} + m_{pq}$. This can be done in $O(n)$ time. Let $P_q$ be the set of all $p$’s corresponding to a column $q$. Then for all $q \in Q$ and $p \in P_q$

\[
\begin{pmatrix}
\ldots \\ m_{ij} & m_{iq} \\
\ldots \\ m_{pj} & m_{pq} \\
\ldots 
\end{pmatrix}
\]

(up to a permutation of rows and columns) is a switchable configuration. This encompasses all the switchable configurations of type $S1$, $S2$ and $S3$, containing $m_{ij}$.

3. For every $q \in Q$ (from step 1), identify pairs of rows $(t, p)$ such that $m_{ij}, m_{tj}, m_{pj}$ are distinct, $m_{iq}, m_{tq}, m_{pq}$ are distinct and $m_{tj} \neq m_{tq}$. This can be achieved in time $O(n^2)$. Let $T_q$ be the set of all such pairs $(t, p)$ corresponding to a column $q$. Then for all $q \in Q$ and
$(t, p) \in T_q$

\[
\begin{bmatrix}
\ldots & \ldots & \ldots & \ldots \\
\ldots & m_{ij} & m_{iq} & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & m_{tj} & m_{tq} & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & m_{pj} & m_{pq} & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

(up to a permutation of rows and columns) is a type-S4 switchable configuration. This step accounts for all switchable configurations of type S4 containing $m_{ij}$.

4. Repeat steps 1, 2 and 3 for all $m_{ij}$. This step has complexity $O(mn)$. 

Since switchings do not change the score sequence, given any $k$-hypertournament matrix $M$ in $\mathcal{M}_S$, we can construct another $k$-hypertournament matrix with the same score sequence in polynomial time provided a switchable configuration exists in $M$. The next result shows that this is always the case.

**Theorem 7.2.4.** Every $k$-hypertournament matrix $M(H)$ has a switchable configuration.

**Proof.** Suppose there is a row in $M(H)$ containing $-1$ in distinct columns $i$ and $j$. Then there is a switchable configuration of type $S1$ in $M(H)$, as otherwise arc $e_i$ is parallel to arc $e_j$, which is not possible.

\[
\begin{bmatrix}
-1 & 1 & \ldots & \ldots & \ldots & \ldots \\
0 & -1 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & 0 & 1 & \ldots & \ldots \\
\ldots & \ldots & -1 & 0 & \ldots & \ldots \\
\ldots & \ldots & 1 & -1 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & -1 & 0 \\
\ldots & \ldots & \ldots & \ldots & 1 & -1 \\
\end{bmatrix}
\]

Figure 7.3: Arrangement of zeros and ones leading to a switchable configuration of type $S4$. 

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If $n > k + 1$, the matrix $M(H)$ is not square. Therefore, there is a row containing two entries equal to $-1$, and thus every non-square $k$-hypertournament matrix contains a switchable configuration.

If $n = k + 1$, $M(H)$ is a square matrix. Suppose no row contains two $-1$'s. Then there is a labelling of columns such that all the $-1$ entries are on the diagonal. It is routine to check that any arrangement of $0$'s and $1$'s preventing the existence of a switchable configuration of type $S_2$ or $S_3$ will use $n - 1$ zeros (see Figure 7.3). Since there can only be

$$n \left[ \binom{n}{k} - \binom{n-1}{k-1} \right] = n \left[ \binom{k+1}{k} - \binom{k}{k-1} \right] = n$$

zeros in all, the one zero entry left cannot prevent the existence of a switchable configuration of type $S_4$. \hfill \Box

Indeed, it is possible to have a $k$-hypertournament matrix that contains one type of switchable configuration but not another. For instance, the matrix $M(H_3)$ in Example 7.2.1 does not have any switchable configuration of type $S_3$, although it has a switchable configuration of every other type.

### 7.3 Random walk on $k$-hypertournament matrices with a given score sequence

In this section, we construct a Markov chain on the set $M_S$ and prove this Markov chain to be ergodic with a uniform stationary distribution. We adopt definitions and notations from [100] and [123].

In order to start a random walk on the set $M_S$ using the switching operations, we must a priori have a $k$-hypertournament matrix with score sequence $S$. Below we give an algorithm that generates a $k$-hypertournament matrix with a given score sequence $S = [s_i]_1^n$. The main idea of the algorithm is inspired by the proof of Theorem 3 in [113] that uses a system of
distinct representatives of a family of sets. Note that $R = [r_i]_i^n$ is the corresponding losing score sequence, where $r_i = \binom{n-1}{i-1}_k - s_i$, for $i = 1, 2, \ldots, n$.

Algorithm 7.3.1.

1. Let $X_1, X_2, \ldots, X_n$ be pairwise disjoint sets with $|X_i| = r_i$, for $i = 1, 2, \ldots, n$. Construct the family $F = \{X_{i_1} \cup X_{i_2} \ldots \cup X_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. This step takes $O(m)$ time as $F$ contains exactly $\binom{n}{k} = m$ elements.

2. Label the elements of $F$ arbitrarily from 1 to $m$. Initialize a matrix $M = [m_{ij}]$ of order $n \times m$ with $m_{ij} = 1$ if the set $X_i$ is selected in the $j^{th}$ element of $F$ and $m_{ij} = 0$ otherwise. This can be done in $O(km)$ time.

3. Determine a system of distinct representatives (SDR) of the family $F$. By the proof of Theorem 7.3.2 below, such an SDR always exists. Finding such an SDR is equivalent to finding a maximum matching of the bipartite graph $B$ having partite sets $F$ and $X = \bigcup_{i=1}^n X_i$, with an edge joining $F \in F$ and $x \in X$ if and only if $x \in F$. Since the bipartite graph $B$ has $|V| = |F| + |X| = 2m$ vertices and at the most $|E| = m^2$ edges, the matching can be determined in $O(\sqrt{|V|}|E|) = O(m^{5/2})$ time by the Hopcroft–Karp algorithm [97].

4. If $x$ is the representative of $F = X_{i_1} \cup X_{i_2} \ldots \cup X_{i_k}$ in the SDR found in step 3, then find the unique set $X_l$ containing $x$, where $l \in \{i_1, i_2, \ldots, i_k\}$. This takes $O(k)$.

5. In the matrix $M$, change $m_{ij}$ to $-1$ if the representative of the $j^{th}$ element of $F$ is in $X_i$ and output $M$. This places exactly one $-1$ in each column of $M$ and leaves $k - 1$ ones in each column. This takes $O(m)$ time.

Algorithm 7.3.1 runs in $O(m^{5/2})$ time, where $m$ is the number of arcs in a $k$-hypertournament with score sequence $S$ (losing score sequence $R$). The following result shows that Algorithm 7.3.1 is indeed correct.

Theorem 7.3.2. If $R = [r_i]_i^n$ is the losing score sequence of a $k$-hypertournament, then Algorithm 7.3.1 outputs a $k$-hypertournament matrix with the losing score sequence $R$. 

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Proof. The proof is on the same lines as the proof of Theorem 1.1.6 given in [113]. We first show that there exists an SDR for the family $\mathcal{F}$. By Hall’s Marriage Theorem, it suffices to prove that the union of any $p$ members of the family $\mathcal{F}$ contains at least $p$ elements. Let $F_1, F_2, \ldots, F_p$ be any $p$ members of $\mathcal{F}$. Suppose $I$ is the set of subscripts of the $X_i’s$ that make up these $F_i’s$, that is, $\bigcup_{i=1}^{p} F_i = \bigcup_{i \in I} X_i$. Clearly, $p \leq \binom{|I|}{k}$, as each $F_i$ is a union of $k$ of the $X_i’s$. Since $[r_i]_1^n$ satisfies Theorem 1.1.6, $\sum_{i \in I} r_i \geq \binom{|I|}{k}$. Now,

$$\left| \bigcup_{i=1}^{p} F_i \right| = \left| \bigcup_{i \in I} X_i \right| = \sum_{i \in I} |X_i| = \sum_{i \in I} r_i \geq \binom{|I|}{k} \geq p$$

and hence, the SDR always exists.

If $x$ is the representative of $X_{i_1} \cup X_{i_2} \ldots \cup X_{i_k}$ in the SDR then there is a unique $X_l$, $l \in \{i_1, i_2, \ldots, i_k\}$, containing $x$ as the $X_i’s$ are pairwise disjoint. We can, therefore, construct a $k$-hypertournament $H$ with the vertex set $\{v_1, v_2, \ldots, v_n\}$ as follows. Every set of $k$ vertices is joined by an arc. The arc containing the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ is oriented with $v_l$ being the last vertex, where $X_l$ is the unique set containing the distinct representative of $X_{i_1} \cup X_{i_2} \ldots \cup X_{i_k}$, while the other vertices are ordered arbitrarily. Clearly, $H$ is a $k$-hypertournament with losing score sequence $R$ and the output $M$ of Algorithm 7.3.1 is the $k$-hypertournament matrix associated with $H$.

Now we define a random walk $\mathcal{M}_{mc}$ on $\mathcal{M}_S$ consisting of the following steps.

**Definition 7.3.3.**

Step 1. Choose the $k$-hypertournament matrix $M(H) = M$ in $\mathcal{M}_S$ obtained by applying Algorithm 7.3.1 to the score sequence $S$.

Step 2. Toss an unbiased coin. If it lands head side up, then stay at $M(H)$. Repeat Step 2. If it lands tail side up, then choose randomly any switchable configuration in $M(H)$ using Algorithm 7.2.3.

Step 3. Move to $M(H')$ if $M(H')$ is obtained from $M(H)$ by switching the switchable
configuration chosen in Step 2.

Step 4. Go back to Step 2.

We introduce some terminology related to finite-state Markov chains. The reader is referred to [101, 123, 156] for a comprehensive treatment of the subject. Let \( \pi \) be a probability distribution over a finite set \( \Omega \). A Markov chain \( \mathcal{M} \) on the set \( \Omega \) is said to be reversible with respect to \( \pi \) if for all \( x, y \in \Omega \), we have

\[
\frac{\pi(x)}{\pi(y)} = \frac{P(y,x)}{P(x,y)},
\]

where \( P(x,y) \) denotes the probability of transition from the state \( x \) to the state \( y \). The Markov chain \( \mathcal{M} \) is then said to obey the detailed balance condition.

A Markov chain \( \mathcal{M} \) over a finite set \( \Omega \) with a symmetric transition matrix can be represented graphically as an undirected graph \( \mathcal{G} \) with vertex set \( \Omega \) and an edge joining vertices \( x \) and \( y \) if the transition probability \( P(x,y) = P(y,x) \) is positive. If \( \mathcal{G} \) is connected, \( \mathcal{M} \) is said to be irreducible. If every vertex of \( \mathcal{G} \) has a loop, then \( \mathcal{M} \) is aperiodic. A Markov chain \( \mathcal{M} \) is said to be ergodic if it is irreducible and aperiodic. We have the following result regarding the equilibrium distribution of a Markov chain [156].

**Lemma 7.3.4.** Any ergodic Markov chain with a symmetric transition matrix has a uniform stationary distribution.

The main result of this section shows that the random walk \( \mathcal{M}_{mc} \) is an ergodic Markov chain on \( \mathcal{M}_S \) with a uniform stationary distribution.

**Theorem 7.3.5.** If \( \mathcal{M}_S \) is the set of all \( k \)-hypertournament matrices with a given score sequence \( S \), then the random walk \( \mathcal{M}_{mc} \) is a symmetric and ergodic Markov chain running on \( \mathcal{M}_S \) with a uniform stationary distribution \( \pi \).

**Proof.** Let \( \mathcal{G} \) be the graph with vertex set \( \mathcal{M}_S \) where two vertices \( M(H) \) and \( M(H') \) are connected by an edge if and only if there is a switching transforming \( M(H) \) into \( M(H') \).
Since, by Lemma 7.2.2, every switching can be reversed, the transition probabilities are symmetric. Therefore, we consider $\mathcal{G}$ as an undirected graph. It suffices to show that $\mathcal{G}$ is a connected graph with every vertex having a loop. Indeed, every $k$-hypertournament matrix $M(H)$ has a loop, since we stay at $M(H)$ with probability $1/2$. For connectivity, suppose that $M(H)$ and $M(H')$ are any two $k$-hypertournament matrices in $\mathcal{M}_S$. We show that there is a sequence of switchings converting $M(H)$ into $M(H')$. We call such a sequence of switchings as the canonical path $\hat{p}$ between $M(H)$ and $M(H')$. Let $m_{ij}$ (respectively, $m'_{ij}$) denote the $(i, j)$-entry of the matrix $M(H)$ (respectively, $M(H')$). Suppose we linearly order the indices such that $(i, j) < (k, j)$ if and only if $i < k$ and, for $j \neq l$, $(i, j) < (k, l)$ if and only if $j < l$. We call this linear ordering the canonical ordering.

The canonical path consists of performing switchings so that we match the entries of $M(H)$ to the entries of $M(H')$, one at a time, in the canonical order. To change an entry $m_{ij}$, we use a sequence of switchings that first produces a suitable switchable configuration containing $m_{ij}$, and then switches $m_{ij}$ into $m'_{ij}$.

Suppose $(p, q)$ is the smallest pair, according to the canonical order, such that $m_{pq} \neq m'_{pq}$. If $q = n$ ($p = n$) then we have matched all entries of row $p$ (column $q$) except the last one. Since by Lemma 1.1.7 and Definition 1.1.8, $M(H)$ and $M(H')$ have the same row (column) sums and the same number of non-zero entries in each row (column), we have a contradiction. Thus, $p \neq n$ and $q \neq n$.

We note that the arguments given below can be mirrored for any particular values of $m_{pq}$ and $m'_{pq}$. Therefore, it suffices to consider the case when $m_{pq} = 1$ and $m'_{pq} = 0$. Note that all entries $m_{ij}$ with $(i, j) < (p, q)$ have already been matched. By Lemma 1.1.7 and Definition 1.1.8, the number of zeros and ones in a row of $M(H)$ is the same as the number of zeros and ones in a row of $M(H')$. So, there exists a column $s$ with $q < s$, $m_{ps} = 0$ and $m'_{ps} = 1$. Similarly, by Lemma 1.1.7, there exists a row $r$ with $p < r$ such that $m_{rq} = 0$ and $m'_{rq} = 1$. If $q = n - 1$, then $m_{rs}$ is necessarily equal to 1, as $m'_{rq} = 1$ and the matrices $M(H)$ and $M(H')$ have the same row sums. Thus, $m_{pq}$ is contained in a switchable configuration of
type $S1$ and we are done. If $q < n - 1$, we assume without loss of generality that $m_{rs} \neq 1$.

We now prove the following claim.

Claim: If $m_{rs} \neq 1$, there exists a sequence of two switchings that makes $m_{pq} = 0$.

Case 1: $m_{rs} = -1$.

Since $m'_{rq} = 1$ but $m_{rq} = 0$, there exists a column, say column $u$, such that $q < u$ and $m_{ru} = 1$. Thus the vertex $v_r$ is contained both in arcs $e_s$ and $e_u$ so that it is the last vertex of arc $e_s$ but not the last vertex of arc $e_u$. Thus, the last vertex of $e_u$ must be different from the last vertex of $e_s$. So there exists a row $t \neq r$ such that either $m_{ts} = 1$ and $m_{tu} = -1$, or $m_{ts} = 0$ and $m_{tu} = -1$. In the former instance there exists a switchable configuration of type $S3$ containing $m_{rs}$ as shown in Figure 7.4. Switching this configuration makes $m_{rs} = 1$ and produces a switchable configuration of type $S1$ containing $m_{pq}$. This proves the claim.

![Matrix](image)

Figure 7.4: The form of matrix $M(H)$ with $m_{ts} = 1$ and $m_{tu} = -1$. The * signs represent the entries that have already been matched with $M(H')$. The bold entries represent the switchable configuration of type $S3$ that arises in this case.

In the latter instance, there exists a row, say row $w$, different from rows $p$ and $t$ such that $m_{ws} = 1$ and $m_{wu} = 0$. We encounter two subcases.

Subcase 1.1: $t \neq p$.

There must exist a switchable configuration of type $S4$ containing $m_{rs}$ but not containing $m_{pa}$ as shown in Figure 7.5. Switching this configuration makes $m_{rs} = 1$, without changing $m_{pa}$, so $m_{pq}$ is contained in an $S1$-type switchable configuration and we are done.
Subcase 1.1: The form of matrix $M(H)$ with $m_{ts} = 0$, $m_{tu} = -1$ and $t \neq p$. The * signs represent the entries that have already been matched with $M(H')$. The bold entries represent the switchable configuration of type $S4$ that arises in this case.

Subcase 1.2: $t = p$.

This situation is illustrated in Figure 7.6. Three possibilities arise based on the three possible values of the entry $m_{wq}$ (represented by the □ sign in Figure 7.6).

If $m_{wq} = 0$, we have a switchable configuration of type $S1$ containing $m_{pq}$ and $m_{wq}$. Switching this configuration sets $m_{pq} = 0$. If $m_{wq} = -1$, there must exist a switchable configuration of type $S4$ containing $m_{pq}$ and $m_{wq}$, and once again switching this configuration results in $m_{pq} = 0$. Finally, if $m_{wq} = 1$, we have an $S1$-type switchable configuration across columns $q$ and $u$. The sequence of switchings shown in Figure 7.7 results in $m_{pq} = 0$, and the claim is proved.

Case 2: $m_{rs} = 0$.

Since $m'_{rq} = 1$ but $m_{rq} = 0$, there exists a column, say column $u$, such that $q < u$ and $m_{ru} = 1$. Thus, vertex $v_r$ is not contained in arc $e_s$ but is contained in arc $e_u$, though not
Figure 7.7: Sequence of switchings to obtain $m_{pq} = 0$ when $m_{wq} = 1$. Each switchable configuration is represented by bold entries.

as the last vertex. There necessarily exists a vertex $v_t$, with $t \neq p$ and $t \neq r$, that is not contained in arc $e_u$ but is contained in arc $e_s$, though not as the last vertex. So, there exists a row $t \neq p, r$ such that $m_{ts} = 1$ and $m_{tu} = 0$. This gives a switchable configuration of type $S1$ containing $m_{rs}$ as shown in Figure 7.8. Switching this configuration makes $m_{rs} = 1$ and produces a switchable configuration of type $S1$ containing $m_{pq}$.

Figure 7.8: Case 2: The form of matrix $M(H)$ with $m_{rs} = 0$. The * signs represent the entries that have already been matched with $M(H')$. The bold entries represent the switchable configurations of type $S1$ containing $m_{rs}$ that arise in this case.

This completes the proof of the claim. Thus, every entry of $M(H)$ can be converted into
the corresponding entry of $M(H')$ by a sequence of at most two switchings. We conclude that there exists a canonical path $\hat{p}$ joining any two $k$-hypertournament matrices in $\mathcal{M}_S$, and hence, $\mathcal{G}$ is connected. Furthermore, by Lemma 7.3.4, the stationary distribution $\pi$ is uniform.

The following result gives an upper bound for the number of switchings to match $M(H)$ and $M(H')$, the proof of which follows from Theorem 7.3.5.

**Corollary 7.3.6.** If $M(H)$ and $M(H')$ are two $n \times m$ $k$-hypertournament matrices contained in $\mathcal{M}_S$, then the number of switchings to match $M(H)$ and $M(H')$, using the canonical path, is at most $2nm$.

**Proof.** Any canonical path needs to match at most $nm$ entries of the matrix, and matching any entry requires a sequence of at most 2 switchings.

---

### 7.4 Mixing time of $\mathcal{M}_{mc}$

We view the Markov chain $\mathcal{M}_{mc}$ as running on a network $\vec{G}$ whose vertices are members of $\mathcal{M}_S$ and there is an arc (directed edge) $e = (x, y)$ if there is a switching that converts the $k$-hypertournament matrix $x$ into $y$, i.e., the transition probability from $x$ to $y$, denoted by $P(x, y)$ is positive. Evidently, there is at most one arc directed from $x$ to $y$, since if $x$ can be converted into $y$ by a switching then the switching is unique. Intuitively, the aim of this section is to show that the network $\vec{G}$ has no bottleneck in the sense that there are no cuts between any set of vertices $U$ to its complement that block the flow of the Markov chain, and thus prevent the Markov chain from mixing rapidly. To make this notion more precise we need some preliminary definitions from [101, 123, 156].

Sinclair [156] defined the concept of mixing time to measure the rate at which a Markov chain converges to the stationary distribution.
Definition 7.4.1. Suppose a Markov chain starts in state $x$ and let $\Delta_x(t)$ denote the distance of the Markov chain from the stationary distribution $\pi$ at time $t$. The mixing time of the chain is defined as

$$\tau_x(\epsilon) = \min\{t : \Delta_x(t') \leq \epsilon \text{ for all } t' \geq t\},$$

i.e., the time to reduce the variation distance to $\epsilon$.

If $\pi$ is the stationary distribution, the capacity of $e = (x, y)$, denoted by $c(e)$, is given by

$$c(e) = \pi(x)P(x, y).$$

Since every switching is reversible, to every directed arc $e = (x, y)$, there corresponds another arc $e^{-1} = (y, x)$ such that $c(e) = c(e^{-1})$.

Let $P_{x,y}$ denote the set of all simple paths from $x$ to $y$, and let $P_e$ be the set of all simple paths containing arc $e$. A flow in $\vec{G}$ is a function $\phi$ from the set of simple paths to the real numbers such that

$$\sum_{p \in P_{x,y}} \phi(p) = \pi(x)\pi(y),$$

for all vertices $x$ and $y$ of $\vec{G}$ with $x \neq y$. A flow along an arc $e$ is then defined as

$$\phi(e) = \sum_{p \in P_e} \phi(p).$$

For a flow $\phi$, a measure of existence of an overload along an arc is given by the quantity $\rho(e)$ where

$$\rho(e) = \frac{\phi(e)}{c(e)},$$

and the cost of the flow $\phi$, denoted by $\rho(\phi)$ is

$$\rho(\phi) = \max_e \rho(e).$$
If a network $\vec{G}$ representing a Markov Chain can support a flow of low cost, it cannot have any bottlenecks, and hence, its mixing time should be small. This intuition is confirmed by the following result.

**Theorem 7.4.2 ([156]).** Let $\mathcal{M}$ be an ergodic reversible Markov chain with holding probabilities $P(x,x) \geq \frac{1}{2}$ at all states $x$. Then the mixing time of $\mathcal{M}$ satisfies

$$\tau_x(\epsilon) \leq 8(\rho(\phi))^2 \left(\ln \frac{1}{\pi(x)} + \ln \frac{1}{\epsilon}\right).$$

In order to make use of Theorem 7.4.2 we have to define a flow of low cost for $\mathcal{M}_{mc}$. To do this, first assume that for all pairs $(x,y)$ of vertices of $\vec{G}$, we channel a particular commodity, called $\langle xy \rangle$. We want the commodity $\langle xy \rangle$ to be different from commodities for other pairs so that the commodities do not get mixed up. We also want to channel all the $\pi(x)\pi(y)$ units of commodity $\langle xy \rangle$ along the canonical path $\hat{p}$ between $x$ and $y$ so that

$$\phi(\hat{p}) = \pi(x)\pi(y),$$

and

$$\phi(p) = 0$$

for all paths $p$ between $x$ and $y$ such that $p \neq \hat{p}$.

Now assume that $\vec{G}$ has $N$ vertices with minimum (undirected) degree $d_1$ and maximum degree $d_2$. Furthermore, assume that the length of any canonical path is at most $M$. With these assumptions, we have the following observations.

- $\sum_{x \neq y} \pi(x)\pi(y) \leq 1$, i.e., the total quantity of commodities flowing through the network is at most one unit.

- $\sum_{e} \phi(e) \leq M$, i.e., the total flow along all the arcs is at most $M$.

- By Corollary 7.3.6, we have $M \leq 2mn$, where $n$ is the number of vertices and $m$ is the
number of arcs of a $k$-hypertournament with score sequence $S$.

Since the total number of arcs in $\vec{G}$ is at least $\frac{1}{2}d_1N$, it follows that for all arcs $e$,

$$\phi(e) \leq \frac{2M}{d_1N}. \quad (7.4)$$

**Lemma 7.4.3.** If $\rho$, $\phi$ and $M$ are as defined above, then $\rho(\phi) \leq 4mn$.

**Proof.** For all arcs $e = (x, y)$, we have $\frac{1}{d_2} \leq P(x, y) \leq \frac{1}{\pi}$, $\pi(x) = \frac{1}{N}$ and therefore, $c(e) = \pi(x)P(x, y) \geq \frac{1}{N \cdot d_2}$. Thus, by (7.4), $\rho(e) = \frac{\phi(e)}{c(e)} \leq 2M$. The result follows as $M \leq 2mn$. \qed

To show that the Markov chain $\mathcal{M}_{mc}$ is rapidly mixing we prove that the quantity $\tau_x(\epsilon)$ is bounded above by a polynomial of the network parameters, which in our case are $k$, $m$ and $n$.

**Theorem 7.4.4.** The mixing time of $\mathcal{M}_{mc}$ satisfies

$$\tau_x(\epsilon) \leq 128m^3n^2 \left( \ln k + \ln \frac{1}{\epsilon} \right).$$

**Proof.** By Theorem 7.4.2 and Lemma 7.4.3, we have

$$\tau_x(\epsilon) \leq 128(mn)^2 \left( \ln \frac{1}{\pi(x)} + \ln \frac{1}{\epsilon} \right).$$

Now by Theorem 7.3.5, $\pi$ is uniform so that $\ln \frac{1}{\pi(x)} = \ln N$, where $N$ is the number of vertices in $\vec{G}$, that is, the number of $k$-hypertournament matrices in $\mathcal{M}_S$. Note that there are $k! \binom{n}{k}$ labelled $k$-hypertournaments with $n$ vertices, and by Lemma 7.5.1 in Section 7.5, a given $k$-hypertournament matrix corresponds to $\binom{k}{k-1}! \binom{n}{k}$ of these hypertournaments. Thus,

$$N = \frac{|\mathcal{H}_S|}{(k-1)! \binom{n}{k}} \leq \frac{k! \binom{k}{k}}{(k-1)! \binom{n}{k}} = k^m = k^m,$$
where \( m = \binom{n}{k} \) is the number of arcs in a \( k \)-hypertournament in \( \mathcal{H}_S \). The result follows immediately. \( \square \)

For fixed \( k \), the mixing time of \( \mathcal{M}_mc \) is bounded by \( 128n^{3k+2}(\ln k + \ln \frac{1}{\epsilon}) \), which is a polynomial in \( n \).

### 7.5 The Markov chain on \( k \)-hypertournaments

We now turn our attention to the set \( \mathcal{H}_S \) of all \( k \)-hypertournaments with a given score sequence \( S \). Our aim is to sample uniformly through \( \mathcal{H}_S \) using the uniform sampling of \( \mathcal{M}_S \) performed in Sections 3 and 4. Indeed, we shall construct a Markov chain \( \mathcal{H}_{mc} \) on the set \( \mathcal{H}_S \) using the Markov chain \( \mathcal{M}_{mc} \), and show that \( \mathcal{H}_{mc} \) inherits the nice properties of \( \mathcal{M}_{mc} \). In order to formally define \( \mathcal{H}_{mc} \), we need the following auxiliary result.

#### Lemma 7.5.1

If \( M \) is a \( k \)-hypertournament matrix in \( \mathcal{M}_S \), then there are exactly \( (k-1)!^m \) \( k \)-hypertournaments in \( \mathcal{H}_S \) whose incidence matrix is \( M \).

**Proof.** There are \( (k-1) \) entries equal to 1 in every column of \( M \). We can obtain a \( k \)-hypertournament corresponding to \( M \) and having score sequence \( S \) by ordering the 1’s in each column of \( M \) from 1 to \( k-1 \). Since \( M \) has \( \binom{n}{k} = m \) columns, this can be done in \( (k-1)!^m \) different ways. \( \square \)

We observe that the proof of Lemma 7.5.1 gives an algorithm for constructing a \( k \)-hypertournament corresponding to a \( k \)-hypertournament matrix \( M \). In each column, order the 1’s arbitrarily from top to bottom. This takes \( O(km) \) steps, where \( m \) is the number of columns in \( M \).

We define the Markov chain \( \mathcal{H}_{mc} \) as follows.

#### Definition 7.5.2

Let \( \bar{G} \) be defined as earlier, that is, \( \bar{G} \) is the graph on which the random walk \( \mathcal{M}_{mc} \) is defined. Let us construct a graph \( \bar{G}' \) with vertex set \( \mathcal{H}_S \) such that if \( H_1 \) and \( H_2 \) are two \( k \)-hypertournaments in \( \mathcal{H}_S \) corresponding to \( k \)-hypertournament matrices \( M_1 \) and
Then there is an arc in $\vec{G}$ from $H_1$ to $H_2$ if and only if there is an arc in $\vec{G}$ from $M_1$ to $M_2$. The Markov chain $\mathcal{H}_{mc}$ runs on the graph $\vec{G}$ as follows.

- If at a given time the Markov chain $\mathcal{M}_{mc}$ is at $M$, then $\mathcal{H}_{mc}$ is at $H$, where $H$ is one of the $(k-1)!^m$ hypertournaments corresponding to $M$, chosen uniformly at random.

- If at the next step $\mathcal{M}_{mc}$ stays at $M$, $\mathcal{H}_{mc}$ moves to $H'$, where $H'$ is again one of the $(k-1)!^m$ hypertournaments corresponding to $M$, chosen uniformly at random, (this ensures a positive holding probability) or

- If at the next step $\mathcal{M}_{mc}$ moves to a new state $M'$, $\mathcal{H}_{mc}$ moves to $H'$, where $H'$ is a hypertournament corresponding to $M'$, chosen uniformly at random.

We now show that $\mathcal{H}_{mc}$ inherits all the necessary properties of $\mathcal{M}_{mc}$ so that we can sample uniformly through all $k$-hypertournaments with a given score sequence.

**Theorem 7.5.3.** The Markov chain $\mathcal{H}_{mc}$ is rapidly mixing and ergodic with a uniform stationary distribution.

**Proof.** Note that for any two states ($k$-hypertournaments) $x$ and $y$ of $\mathcal{H}_{mc}$ we have $P(x, y) = P(y, x)$, that is, the transition probabilities are symmetric, and thus we may consider $\vec{G}'$ to be an undirected graph $G'$. Since the holding probability is positive for all states of $\mathcal{H}_{mc}$, $G'$ has a loop at each vertex. The connectedness of $G'$ follows from the connectedness of $G$ (as in Theorem 7.3.5). Thus $\mathcal{H}_{mc}$ is ergodic. By Lemma 7.3.4, $\mathcal{H}_{mc}$ has a uniform stationary distribution. The Markov chain $\mathcal{H}_{mc}$ is also rapidly mixing as $\mathcal{H}_{mc}$ achieves a uniform stationary distribution as soon as $\mathcal{M}_{mc}$ does.

□
Chapter 8

Conclusion and future work

In this thesis, we investigate several algorithmic questions on different classes of graphs (and hypergraphs), and problems arising from discrete arrangements of convex bodies.

We introduce the covering index of convex bodies to quantify how economically a convex body can be covered by few and relatively small positive homothetic copies. We show that the covering index possesses several nice properties such as lower semicontinuity, and compatibility with direct and Minkowski sums, making it tractable to compute the covering index exactly for infinitely many convex bodies. Moreover, we characterize the minimum value of the covering index in all dimensions. However, the following questions still remain unanswered.

**Problem 8.0.1.** Either prove that $\text{coin}(\cdot)$ is upper semicontinuous on $K^d$ or construct a counterexample.

**Problem 8.0.2.** Let $K_1, \ldots, K_n$ be $d$-dimensional convex bodies, for some $d \geq 2$. Then prove (disprove) that

$$\max\{\text{coin}(K_i) : i = 1, \ldots, n\} \leq \text{coin}(K_1 + \cdots + K_n).$$
If this does not hold, one can try proving the following weaker lower bound.

$$\min\{\text{coin}(K_i) : i = 1, \ldots, n\} \leq \text{coin}(K_1 + \cdots + K_n).$$

Like much of the research on homothetic covering of convex bodies, our investigation is fuelled by the Hadwiger Covering Conjecture and we propose an approach based on the covering index to improve the current best upper bound in connection with this conjecture.

**Problem 8.0.3.** For any $d$-dimensional convex body $K$, prove or disprove that $\text{coin}(K) \leq \text{coin}(B^d)$.

An affirmative answer to the above problem would exponentially improve the known best general upper bound on the illumination number as discussed in Chapter 3.

We then study the Hadwiger and contact numbers of totally separable packings of translates of convex bodies. These numbers arise, respectively, as the maximum degree and total number of edges in contact graphs of totally separable translatative packings of a convex body, and are of interest from the point of views of geometry, crystallography and materials science. We determine the separable Hadwiger number of any smooth convex domain, and the maximum separable contact number of $n$ translates of a smooth strictly convex domain for any $n$. In addition, we show that the configurations achieving the separable Hadwiger number as well as the ones maximizing the separable contact numbers exhibit crystalline order. In fact, these extremal configurations can be thought of as totally separable ground-states of sticky pair-potentials in normed spaces. We propose to investigate the following extensions of our results.

**Conjecture 8.0.4.** If $K$ is a smooth convex domain, then $c_{\text{sep}}(K, n, 2) = \lfloor 2n - 2\sqrt{n} \rfloor$.

**Problem 8.0.5.** If $K$ is a smooth convex body in $E^d$, then is $H_{\text{sep}}(K) = 2d$?

**Problem 8.0.6.** Determine or find good estimates for $c_{\text{sep}}(\cdot, n, 2)$ and $c_{\text{sep}}(\cdot, n, d)$, for convex domains and $d$-dimensional convex bodies, respectively. Do maximal contact configurations always exhibit periodic order?
The discussion then moves from the more geometric questions on contact graphs to the more combinatorial questions on families of directed graphs and hypergraphs. The families we consider include tournaments and hypertournaments. In particular, we are interested in generating tournaments from their imbalance sets and hypertournaments from their score sequences.

We characterize the imbalance sets of tournaments and give an algorithm to construct tournaments with a given imbalance set. Interestingly, we discover that the Tournament Imbalance Set problem has a strong connection with the Equal-Sum Subsets problem from computer science. We study this connection and use it as a basis for our algorithm. Since our algorithm runs in exponential time, we propose to look for ways to reduce the computational complexity.

**Problem 8.0.7.** Is it possible to speed up Algorithm 6.4.2? In particular, can a faster procedure be developed to replace Equal Seq subroutine of Algorithm 6.4.2?

Furthermore, the following question can be asked about the complexity of the Tournament Imbalance Set (TIS).

**Problem 8.0.8.** Is the decision (search) version of TIS NP-complete (NP-hard)?

Finally, we characterize regularity of hypertournaments by three different results including one involving the maximum number of directed paths of length 2 in a hypertournament. We then propose a rapidly mixing Markov chain Monte Carlo (MCMC) procedure to uniformly generate hypertournaments having a given score sequence. Thus we solve the “degree sequence problem” for hypertournaments. Our approach is based on combinatorial matrix theory and the following is a natural question to consider.

**Problem 8.0.9.** Can we adapt the combinatorial matrix theoretic techniques introduced in this thesis to construct a rapidly mixing Markov chain to uniformly sample simple graphs (hypergraphs) with a given degree sequence?
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