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Spindle starshaped sets

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Abstract. In this paper, spindle starshaped sets are introduced and investigated, which apart from normalization form an everywhere dense subfamily within the family of starshaped sets. We focus on proving spindle starshaped analogues of recent theorems of Bobylev, Breen, Toranzos, and Zamfirescu on starshaped sets. Finally, we consider the problem of guarding treasures in an art gallery (in the traditional linear way as well as via spindles).

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1. Introduction

We denote the origin of a Euclidean d-space $\mathbb{E}^d$ by o, a closed (resp., open) Euclidean ball in $\mathbb{E}^d$ centered at z of radius λ by $\mathbf{B}[z,\lambda]$ (resp., $\mathbf{B}(z,\lambda)$), its boundary, the sphere by $\mathbb{S}(z,\lambda)$. When λ is omitted, it is 1. For the *circumradius* of a set $A \subseteq \mathbb{E}^d$, we use $\operatorname{cr} A = \inf\{r > 0 \ : \ A \subseteq \mathbf{B}[q,r] \text{ for some } q \in \mathbb{E}^d\}$. We use the usual notations, $\operatorname{int} A, \operatorname{bd} A, \operatorname{conv} A$ for the interior, the boundary and the convex hull, respectively, of a set A in $\mathbb{E}^d$. If A is a subset of some unit sphere $\mathbb{S}(p)$, we denote its spherical convex hull by $\operatorname{Sconv} A$.

For any $\lambda > 0$ we define the $\lambda-$*spindle* of two points $x, y \in \mathbb{E}^d$ as

$$[x,y]_{\mathrm{s}}^{\lambda} = \bigcap\{\mathbf{B}[u,\lambda] \ : \ u \in \mathbb{E}^d; x,y \in \mathbf{B}[u,\lambda]\}$$

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if $|x - y| \leq 2\lambda$ (with $|\cdot|$ standing for the standard norm of $\mathbb{E}^d$), and as $[x, y]_{\mathrm{s}}^{\lambda} = \mathbb{E}^d$ otherwise. Unless otherwise noted, $\lambda = 1$, and λ is omitted. Clearly, for any $\lambda < \nu$ we have $[x, y]_{\mathrm{s}}^{\nu} \subset [x, y]_{\mathrm{s}}^{\lambda}$.

We recall that for a set $A \subseteq \mathbb{E}^d$ and a point $p \in A$, the *visibility region* of p in A is

$$\mathrm{st}\,(p, A) = \{q \in A \ : \ [p, q] \subseteq A\},$$

where $[p, q]$ refers to the line segment joining p and q. When obvious from the context, we may omit A. The *kernel* of A is $\ker A = \{p \in A \ : \ \mathrm{st}\,(p, A) = A\}$. We say that A is a *starshaped set* if $\ker_{\mathrm{s}} A \neq \emptyset$. In particular, a starshaped set is non-empty.

We define the spindle analogues of these notions. For a set $A \subseteq \mathbb{E}^d$ and a point $p \in A$, the *spindle visibility region* of p in A is

$$\mathrm{st}_{\mathrm{s}}\,(p, A) = \{q \in A \ : \ [p, q]_{\mathrm{s}} \subseteq A\}.$$

When obvious from the context, we may omit A. The *spindle kernel* of A is $\ker_{\mathrm{s}} A = \{p \in A \ : \ \mathrm{st}_{\mathrm{s}}\,(p, A) = A\}$. We say that A is a *spindle starshaped set* if $\ker_{\mathrm{s}} A \neq \emptyset$. In particular, a spindle starshaped set is non-empty.

We recall from [1] that A is called *spindle convex* if $A = \ker_{\mathrm{s}} A$. Note that if $\mathrm{cr}\, A > 1$ (resp., $\mathrm{cr}\, A > 2$), then A is spindle convex (or spindle starshaped) if, and only if, $A = \mathbb{E}^d$. We use $\mathrm{conv}_{\mathrm{s}}\,(\cdot)$ to denote the spindle convex hull of a set, i.e., the intersection of all spindle convex sets containing the given set. As we will see (Corollary 6.1), if S is a spindle starshaped set in $\mathbb{E}^d$, then its spindle kernel $\ker_{\mathrm{s}}(S)$ is spindle convex.

A motivation for the study of spindle starshaped sets is that a star shaped set with C^2 boundary whose curvature is bounded away from zero is necessarily spindle starshaped with respect to λ-spindles for a sufficiently large λ.

Krasnosselsky's well-known theorem [13] states the following: Let S be a compact set in $\mathbb{E}^d$. Assume that for any $d+1$ points $x_1, \ldots, x_{d+1}$ of S, there is a point $y \in S$ such that $\cup_{i=1}^{d+1} [y, x_i] \subset S$. Then S is a starshaped set. Bobylev [2,3] observed that the same proof gives the following somewhat stronger result: Let S be a compact set in $\mathbb{E}^d$. Then the kernel of S can be obtained as

$$\ker S = \bigcap_{x \in S} \mathrm{conv}\,\big(\mathrm{st}(x, S)\big). \tag{1}$$

We prove the following (somewhat stronger) spindle analogue of (1).

Theorem 1.1. *Let S be a compact set in $\mathbb{E}^d$. Then the spindle kernel of S can be obtained as*

$$\ker_{\mathrm{s}} S = \bigcap_{x \in \mathrm{bd}\, S} \mathrm{conv}_{\mathrm{s}}\,(\mathrm{st}_{\mathrm{s}}\,(x, S))\,.$$

This theorem combined with Helly's theorem [7,11] yields

Corollary 1.1. *Let S be a compact set in $\mathbb{E}^d$. Assume that for any $d+1$ points $x_1,\ldots,x_{d+1}$ of* $\operatorname{bd} S$*, there is a point $y \in S$ such that $\cup_{i=1}^{d+1}[y,x_i]_{\mathrm{s}} \subset S$. Then S is a spindle starshaped set.*

Theorem 1.1 combined with Klee's version of Helly's theorem [12] yields the following

Corollary 1.2. *Let S and K be compact sets in $\mathbb{E}^d$. Assume that for any $d+1$ points $x_1,\ldots,x_{d+1}$ of* $\operatorname{bd} S$*, there is a vector $u \in \mathbb{E}^d$ such that $u + K \subseteq \bigcap_{i=1}^{d+1} \operatorname{conv}_{\mathrm{s}}(\operatorname{st}_{\mathrm{s}}(x_i,S))$. Then there is a vector $v \in \mathbb{E}^d$ such that $v+K \subseteq \ker_{\mathrm{s}} S$.*

The next theorem is a discrete relative of Theorem 1.1. It is based on sets called *flowers* that have been introduced by Gordon and Meyer [10] and have been studied by Csikos [6] as well from the point of view of the Kneser–Poulsen conjecture. Here, we need the following version of flowers. Let F be a *lattice polynomial*, i.e., an expression built up from some variables using the binary operations $\cap$ and $\cup$ with properly placed brackets indicating the order of the evaluation of the operations. We identify two lattice polynomials if they can be obtained from one another using the commutativity and associativity of the operations $\cap$ and $\cup$. (Other lattice identities are not used in the identification). Also, it will be convenient to write F as $F(x_1,\ldots,x_n)$ indicating the variables of F by $x_1,\ldots,x_n$. We shall assume that each variable x_i occurs exactly once in F. A *flower-polyhedron* in $\mathbb{E}^d$ is a set of the form $F(B_1,\ldots,B_n)$, where F is a lattice polynomial and the sets $B_1,\ldots,B_n$ are closed unit balls in $\mathbb{E}^d$. Finally, a flower-polyhedron in $\mathbb{E}^d$ is called *reduced along its boundary* if the boundary of the flower-polyhedron intersects the boundary of each generating unit ball in a $(d-1)$-dimensional set.

Theorem 1.2. *Let $F(B_1,\ldots,B_n)$ be a flower-polyhedron reduced along its boundary in $\mathbb{E}^d$. Then* $\ker_{\mathrm{s}}(F(B_1,\ldots,B_n)) = \bigcap_{i=1}^{n} B_i$.

Clearly, Theorem 1.2 leads to a geometric construction of the family of flower-polyhedra with d-dimensional spindle kernels in $\mathbb{E}^d$. The set $\mathcal{S}$ of all compact spindle starshaped sets in $\mathbb{E}^d$ equipped with the Hausdorff distance is a Baire space by Baire's Category Theorem, since it is a complete metric space. Now, recall that a property is called *typical* for a member of a Baire space, if the set of those members that do not have the property is of category one, i.e., they are a union of countably many nowhere dense sets. Although it is easy to see that flower-polyhedra with d-dimensional spindle kernels in $\mathbb{E}^d$ form an everywhere dense set in $\mathcal{S}$, still the following theorem holds, which in fact, is a spindle analogue of a theorem of Zamfirescu [18] claiming that the kernel of a typical starshaped set is a singleton.

Theorem 1.3. *The spindle kernel of a typical compact spindle starshaped set of $\mathbb{E}^d, d \geq 2$ is a singleton.*

Next, we prove an analogue of a result of Toranzos and Forte Cunto [9] that characterizes locally kernel points of starshaped sets. Let $S \subset \mathbb{E}^d$ be a compact set. A point x of S is a *spindle peak* of S if there is a neighborhood U of x such that for any $x' \in U$ we have $\operatorname{st_s}(x') \subseteq \operatorname{st_s}(x)$.

Theorem 1.4. *Let $S \subset \mathbb{E}^d$ be a compact set such that $\operatorname{cl}(\operatorname{int} S) = S$ and $\operatorname{int} S$ is connected. Assume that $x \in S$ is a spindle peak of S. Then $x \in \ker_s S$ (and hence, S is necessarily spindle starshaped).*

Recently Bobylev [2,3] proved an elegant version of Helly's theorem for starshaped sets: Let $S_i \ (i \in I)$ be a family of compact starshaped sets in $\mathbb{E}^d$ with $\operatorname{card} I \geq d+1$. Assume that for every $i_1, \ldots, i_{d+1} \in I$ the intersection $\cap_{j=1}^{d+1} S_{i_j}$ is starshaped. Then the intersection $\cap_{i \in I} S_i$ is starshaped. Breen [4] has strengthened that result as follows: Let $S_i \ (i \in I)$ be a family of compact starshaped sets with $\operatorname{card} I \geq d+1$ and K be a compact set in $\mathbb{E}^d$. Assume that for every $i_1, \ldots, i_{d+1} \in I$ there is a vector $u \in \mathbb{E}^d$ such that $u + K \subseteq \ker(\cap_{j=1}^{d+1} S_{i_j})$. Then there is a vector $v \in \mathbb{E}^d$ such that $v + K \subseteq \ker(\cap_{i \in I} S_i)$. We prove the following spindle analogue of these results.

Theorem 1.5. *Let $S_i \ (i \in I)$ be a family of compact spindle starshaped sets with $\operatorname{card} I \geq d+1$ and K be a compact set in $\mathbb{E}^d$. Assume that for every $i_1, \ldots, i_{d+1} \in I$ there is a vector $u \in \mathbb{E}^d$ such that $u + K \subseteq \ker_s(\cap_{j=1}^{d+1} S_{i_j})$. Then there is a vector $v \in \mathbb{E}^d$ such that*

$$v + K \subseteq \ker_s\left(\bigcap_{i \in I} S_i\right).$$

In [5], Breen proves that if every countable subfamily of a family $\mathcal{F}$ of starshaped sets (not necessarily compact) has a (non-empty) starshaped intersection, then the intersection of $\mathcal{F}$ is also starshaped. The analogous statement for spindle starshaped sets follows. The proof in our setting—since spindle starshaped sets are "fat"—is somewhat simpler than Breen's. The main idea is to study the trace of our set family on $\mathbb{Q}^d$.

Theorem 1.6. *Let K be a set in $\mathbb{E}^d$ and $\mathcal{F}$ be a family of spindle starshaped sets in $\mathbb{E}^d$ with the property that the intersection of any countable subfamily of $\mathcal{F}$ is a spindle starshaped set whose spindle kernel contains a translate of K. Then $\ker_s \bigcap \mathcal{F}$ also contains a translate of K.*

Finally, we consider the problem of guarding only certain points in a planar art gallery—a question known as "Treasures in an art gallery" in computational geometry [17]. We characterize the case when a single guard suffices. We prove both a linear and a spindle version of the result. To our knowledge, both have been unknown.

Theorem 1.7. *Let S be a compact, simply connected set in the plane and A be some finite subset of S. Assume that for any three points of A there is a point* *in S from which each one is visible within S. Then there is a point in S that can see all points of A.*

Theorem 1.8. *Let S be a compact, simply connected set of diameter at most 2 in the plane and A be some finite subset of S. Assume that for any three points of A there is a point in S from which each one is visible within S via a spindle. Then there is a point in S that can see all points of A via spindles.*

Neither the planarity nor the simple connectedness condition can be dropped, see Remark 9.1.

In the rest of the paper we prove the stated theorems.

2. Krasnosselsky-type result: proof of Theorem 1.1

In what follows, an *arc* (or *circular arc*) is a connected subset of a circle, which contains no pair of antipodal points of the circle.

Clearly, $\ker_{\mathrm{s}} S \subseteq \bigcap_{x\in \mathrm{bd}\, S} \mathrm{conv}_{\mathrm{s}}\left(\mathrm{st}_{\mathrm{s}}\left(x,S\right)\right)$. So, we are left to show that

$$\bigcap_{x\in \mathrm{bd}\, S} \mathrm{conv}_{\mathrm{s}}\left(\mathrm{st}_{\mathrm{s}}\left(x,S\right)\right) \subseteq \ker_{\mathrm{s}} S.$$

We may assume that

$$\bigcap_{x\in \mathrm{bd}\, S} \mathrm{conv}_{\mathrm{s}}\left(\mathrm{st}_{\mathrm{s}}\left(x,S\right)\right) \neq \emptyset,$$

as otherwise there is nothing to prove. We pick a point y_0 from this intersection. Note that

$$S \subseteq \mathbf{B}[y_0, 2]. \tag{2}$$

Indeed, let z be a point of S that is furthest from y_0. Suppose for a contradiction that $r := |y_0 - z| > 2$. Then $S \subseteq \mathbf{B}[y_0, r]$, and hence $\mathrm{st}_{\mathrm{s}}\left(z\right)$ is in the closed unit ball that touches $\mathbb{S}[y_0, r]$ at z from inside. It follows that $\mathrm{conv}_{\mathrm{s}}\left(\mathrm{st}_{\mathrm{s}}\left(z\right)\right)$ is in this unit ball, too, and thus does not contain y_0, which is a contradiction.

In three steps we will show that y_0 is a spindle star center of S.

Step 1. We show that for any $x \in S$ we have $[x, y_0]_{\mathrm{s}}^{\sqrt{2}} \subseteq S$. In this step, we follow closely Krasnosselsky's proof from [13].

Suppose, for a contradiction, that there is $x_0 \in S$ such that $[x_0, y_0]_{\mathrm{s}}^{\sqrt{2}} \not\subseteq S$.

Then, by the compactness of S, there is an arc γ of radius greater than $\sqrt{2}$ connecting x_0 with y_0 such that some point $x' \in \gamma$ is not in S. For any two points $a, b \in \gamma$, we denote the closed part of γ between a and b by $\gamma[a, b]$ and the open arc by $\gamma(a, b)$.

Let x_1 be the point of $S \cap \gamma[x', x_0]$ that is closest to x' (note that $x_1 \in \mathrm{bd}\, S$ may or may not coincide with x_0). Let x_2 denote a point in $\gamma(x_1, x')$ which is very close to x_1, more precisely such that $d(x_1, x_2) < d(x', S)$. Finally, let x_3

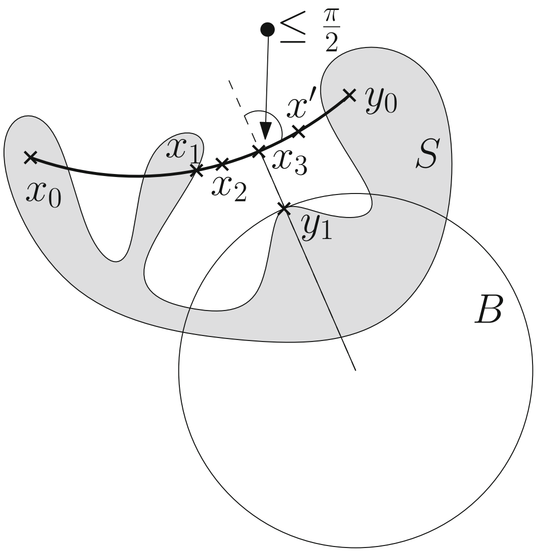

Figure 1. Step 1 in the proof of Theorem 1.1

and $y_1 \in \operatorname{bd} S$ denote a pair of points of the sets $\gamma[x_2, x']$ and S, respectively, at which the distance of the two (compact) sets is attained. Note that x_3 may coincide with x_2, but is certainly not the same as x'. See Fig. 1.

Now, we denote by B the unit ball passing through y_1 with outer normal vector $x_3 - y_1$ at y_1.

On the one hand, we claim that y_0 is not in B. Consider the angle of the vector $x_3 - y_1$ with the direction vector of the non-degenerate arc $\gamma[x_3, y_0]$ (oriented from x_3 toward y_0). From the choice of x_3 and y_1 it follows that this angle is at most a right angle. Since $d(x_0, y_0) \le 2$ (by (2)), and the radius of γ is greater than $\sqrt{2}$, γ is shorter than a quarter circle. Applying Lemma 2.1, we obtain that $y_0 \notin B$.

Lemma 2.1. *Let $z \in \mathbb{E}^d$ and $x \in \mathbb{E}^d \setminus \mathbf{B}[z]$ be given points. Let C be a circle of radius at least one with an arc μ which is shorter than a quarter circle and has end points x and y, oriented from x toward y. Assume that the angle of the vector $x - z$ with μ is at most a right angle. Then $y \notin \mathbf{B}[z]$.*

Proof. Since $\mathbf{B}[z] \cap C$ is an arc of C (which is shorter than a semicircle), $C \setminus \mathbf{B}[z]$ is longer than a semicircle. From the assumption on the angle of μ and $x - z$, it follows that if we consider the same orientation of $C \setminus \mathbf{B}[z]$ as that of μ then x precedes the midpoint of $C \setminus \mathbf{B}[z]$. Since μ is shorter than a quarter circle, $\mu \subset C \setminus \mathbf{B}[z]$. □

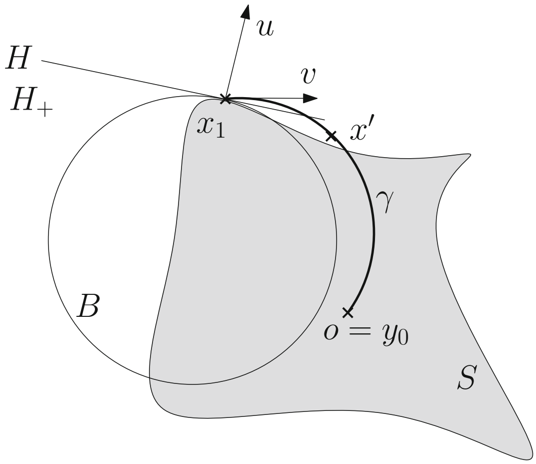

FIGURE 2. Step 2 in the proof of Theorem 1.1

On the other hand, we claim that $y_0 \in B$. Indeed, if a point $z \in \mathbb{E}^d$ is not in B then $[y_1, z]_{\mathrm{s}}$ has points closer to x_3 than y_1, and thus z cannot be in $\mathrm{st}_{\mathrm{s}}(y_1, S)$. That is, $\mathrm{st}_{\mathrm{s}}(y_1, S) \subset B$, and hence, $\mathrm{conv}_{\mathrm{s}}(\mathrm{st}_{\mathrm{s}}(y_1, S)) \subset B$ from which it follows that $y_0 \in B$.

Step 2. A case with a differentiability assumption.

Suppose, for a contradiction, that there is $x_0 \in S \setminus \{y_0\}$ such that for some arc γ of radius $1 < r < \sqrt{2}$ there is a point $x' \in \gamma$ which is not in S. Let x_1 be the point of $S \cap \gamma[x', x_0]$ that is closest to x' (note that $x_1 \in \mathrm{bd}\, S$ may or may not coincide with x_0). We may assume without loss of generality that y_0 is the origin.

Assume that $\mathrm{bd}\, S$ is differentiable at x_1. More precisely, let $f : \mathbb{E}^d \to \mathbb{R} \cup \{\infty\}$ be the *distance function* (or *gauge function*) of S, that is $f(x) = \inf\{\lambda > 0 \ : \ x \in \lambda S\}$. By Step 1, there is a neighborhood of x_1, in which the values of f are all real and not infinity. We assume that f is differentiable at x_1. Let $u = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^d}\right)(x_1)$ denote the gradient of f at the point x_1. Since $f(x_1) = 1 > 0$ and f is *positively homogeneous* (i.e. $f(\mu x) = \mu f(x)$ for any $x \in \mathbb{E}^d$ and $\mu > 0$), we have that u is a non-zero vector.

Let H denote the hyperplane through x_1 with normal vector u, and B denote the unit ball touching H at x_1 with outer normal vector u at x_1. Let H_+ denote the closed halfspace bounded by H and containing B. See Fig. 2.

On the one hand, we claim that y_0 is not in B. The open arc $\gamma(x_1, x')$ is disjoint from S. From the differentiability of f at x_1, it follows that the angle of u and the direction vector (denote it by v) of $\gamma[x_1, y_0]$ at x_1 (we consider γ to be oriented from x_1 toward y_0) is at most a right angle. More precisely, consider a parametrization of γ as follows: $\hat{\gamma} : [0, 1] \to \mathbb{E}^d$, where $\hat{\gamma}(0) = x_1$, and such that $\left(\frac{d}{dt}\hat{\gamma}(t)|_{t=0+}\right) \neq 0$ (here "0+" means right–sided derivative at $t = 0$). Then $0 \leq \frac{d}{dt} f(\hat{\gamma}(t))|_{t=0+}$ since otherwise there would be an $\varepsilon > 0$ such that $f(\hat{\gamma}(\varepsilon)) < f(\hat{\gamma}(0)) = f(x_1) = 1$. By Step 1, S is starshaped and thus, it follows that $\hat{\gamma}(\varepsilon) \in S$, a contradiction. Next,

$$\frac{d}{dt} f(\hat{\gamma}(t))|_{t=0+} = \left(\frac{d}{dt}\hat{\gamma}(t)|_{t=0+}\right) \cdot \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^d}\right)(x_1) = v \cdot u. \tag{3}$$

Since γ is of radius greater than one and $\gamma[x_1, y_0]$ is not longer than a semicircle, it follows that y_0 is not in B.

On the other hand, we claim that $y_0 \in B$. Let $z \in \mathbb{E}^d$ be a point that is not in B. Then $[x_1, z]_{\mathrm{s}}$ contains an arc (call it ω) starting at x_1 that leaves the halfspace H_+. In other words, the angle of the direction vector w of ω (oriented from x_1 toward z) at x_1 and the vector u is acute, that is $w \cdot u > 0$. The same computation as in (3), shows that $\frac{d}{dt} f(\hat{\omega}(t))|_{t=0+} > 0$, where $\hat{\omega}$ is a parametrization of ω such that $\hat{\omega}(0) = x_1$. Hence, there is $\varepsilon > 0$ such that $f(\hat{\omega}(t)) > f(\hat{\omega}(0)) = f(x_1) = 1$ for all $t \in (0, \varepsilon)$. In other words, a non-degenerate open starting section of ω is disjoint from S. Hence, $[x_1, z]_{\mathrm{s}} \not\subset S$ that is, $z \notin \mathrm{st}_{\mathrm{s}}(x_1, S)$. Thus we proved that $\mathrm{st}_{\mathrm{s}}(x_1, S) \subset B$, and hence, $\mathrm{conv}_{\mathrm{s}}(\mathrm{st}_{\mathrm{s}}(x_1, S)) \subset B$. From the definition of y_0, we obtain that $y_0 \in B$.

Step 3. The general case.

Let γ, f and $x_1 \in \mathrm{bd}\, S$ denote the same as in Step 2. By the compactness of $\mathrm{bd}\, S$ it follows that there is a neighborhood U of x_1 such that for any point x_2 in $U \cap \mathrm{bd}\, S$, there is an arc ξ (of radius greater than one) connecting x_2 with y_0 such that a non-degenerated open section of ξ starting at x_2 is disjoint from S. By Step 2, it is sufficient to find a point x_2 in U for which f is differentiable at x_2.

From Step 1 it follows that there is a neighborhood $U_1 \subset U$ of x_1 in which the values of f are all real (and not infinity) and on which f is a Lipschitz function. This is not hard to see. For a similar result on (linearly) starshaped sets with a full–dimensional kernel (which is not necessarily the case here), see [16].

By Rademacher's theorem (cf. Section VII/23 of [8]), f is differentiable almost everywhere in U_1. From the positive homogeneity of f it follows that if f is differentiable at a point $z \in U_1$ then f is differentiable at any point on the ray emanating from the origin and passing through z. Since points of the form $z = \mu x_2$ (where $x_2 \in U_1 \cap \mathrm{bd}\, S$ and $\mu > 0$) are of positive Lebesgue measure in U_1, it follows that there is a point x_2 in $\mathrm{bd}\, S \cap U_1$ where f is differentiable. This concludes the proof of Theorem 1.1.

3. Flowers: proof of Theorem 1.2

First, we show that $\ker_{\mathrm{s}}(F(B_1,\ldots,B_n)) \subseteq \bigcap_{i=1}^{n} B_i$. Indeed, let us assume in an indirect way that there exists a point $x \in \ker_{\mathrm{s}}(F(B_1,\ldots,B_n)) \setminus B_{i_0}$ for some $1 \leq i_0 \leq n$. As $F(B_1,\ldots,B_n)$ is reduced along its boundary in $\mathbb{E}^d$, there exists a point $y \in \operatorname{bd} B_{i_0}$ and an open ball $\mathbf{B}(y,\epsilon)$ centered at y with radius $\epsilon > 0$ such that

$$\mathbf{B}(y,\epsilon) \cap \operatorname{bd} B_{i_0} = \mathbf{B}(y,\epsilon) \cap \operatorname{bd} F(B_1,\ldots,B_n). \tag{4}$$

As $y \in \operatorname{bd} B_{i_0}$ and $x \notin B_{i_0}$, $[y,x]_{\mathrm{s}}$ contains an arc starting at y that leaves B_{i_0}. Clearly, this together with (4) contradicts $x \in \ker_{\mathrm{s}}(F(B_1,\ldots,B_n))$.

Second, we show that $\ker_{\mathrm{s}}(F(B_1,\ldots,B_n)) \supseteq \bigcap_{i=1}^{n} B_i$. It is convenient to prove it by induction on n. The claim is obvious for $n = 1$. So, assume that it holds for any positive integer at most n-1 and write $F(B_1,\ldots,B_n) = G(B_1,\ldots,B_m) \cap H(B_{m+1},\ldots,B_n)$ (resp., $F(B_1,\ldots,B_n) = G(B_1,\ldots,B_m) \cup H(B_{m+1},\ldots,B_n)$) with $1 \leq m \leq n-1$, where G and H are lattice polynomials of at most $n-1$ variables. We note that the flower-polyhedra $G(B_1,\ldots,B_m)$ and $H(B_{m+1},\ldots,B_n)$ are reduced along their boundaries in $\mathbb{E}^d$. Thus, by the inductive assumption

$$\ker_{\mathrm{s}}(G(B_1,\ldots,B_m)) \supseteq \bigcap_{i=1}^{m} B_i, \text{ and } \ker_{\mathrm{s}}(H(B_{m+1},\ldots,B_n)) \supseteq \bigcap_{i=m+1}^{n} B_i. \tag{5}$$

Hence, (5) implies in a straightforward way that

$$\begin{aligned}\ker_{\mathrm{s}}(F(B_1,\ldots,B_n)) &\supseteq \ker_{\mathrm{s}}(G(B_1,\ldots,B_m)) \cap \ker_{\mathrm{s}}(H(B_{m+1},\ldots,B_n)) \\ &\supseteq \bigcap_{i=1}^{n} B_i,\end{aligned}$$

finishing the proof of Theorem 1.2.

4. Typical sets: proof of Theorem 1.3

Denote by $\mathcal{S}_n \subset \mathcal{S}$ the family of those compact spindle starshaped sets, whose kernel contains a ball of radius $\frac{1}{n}$. It is sufficient to prove that $\mathcal{S}_n$ is nowhere dense in $\mathcal{S}$.

Let $S \in \mathcal{S}_n$ be a spindle starshaped set and $\varepsilon > 0$. We will show that in the ε–neighborhood of S, there is $S' \in \mathcal{S}$ such that in some neighborhood of S' there is no element of $\mathcal{S}_n$.

Let $x \in \operatorname{int}\ker_{\mathrm{s}}(S)$ be a point. Take a line ℓ through x, and denote the two endpoints of the non-degenerate line segment $\ell \cap S$ by u and v. Let u' and v' be points on ℓ close to u and v, respectively, but not on the line segment

$[u, v]$. Now we attach two small "spikes" onto S: let $S' = S \cup [x, u']_{\mathrm{s}} \cup [x, v']_{\mathrm{s}}$. We have $\ker_{\mathrm{s}} S' = \{x\}$. Furthermore, clearly, there is $\mu > 0$ such that in the μ–neighborhood of S' in $\mathcal{S}$, no set has a spindle kernel, which contains a ball of radius $\frac{1}{n}$. This finishes the proof.

5. Local characterization: proof of Theorem 1.4

The proof is somewhat simpler than the one in [9]. First, $\operatorname{int}(\mathrm{st}_{\mathrm{s}}(y))$ is not empty for any $y \in \operatorname{int} S$, and $S = \operatorname{cl}(\operatorname{int} S)$ hence, $M = \operatorname{int}(\mathrm{st}_{\mathrm{s}}(x))$ is not empty. Suppose for a contradiction that x is not in the spindle kernel of S and so, $\operatorname{int} S \neq M$. Since $\operatorname{int} S$ is connected and M is open and not empty, it follows that there is a point, say t, in $\operatorname{int} S \cap \operatorname{bd} M$. Let $\varepsilon > 0$ be such that $B = \mathbf{B}[t, \varepsilon] \subseteq \operatorname{int} S$. We have that S contains $B \cup [x, t]_{\mathrm{s}}$. Clearly, for any $\delta > 0$ we can find a point $x' \in \mathbf{B}[x, \delta]$ on (x, t) and t' close to t on the ray emanating from t in the direction $\overrightarrow{xt}$ such that $[x', t']_{\mathrm{s}} \subset B \cup [x, t]_{\mathrm{s}}$. Since $t \in \operatorname{int}[x', t']_{\mathrm{s}}$, we have that $\mathrm{st}_{\mathrm{s}}(x)$ does not contain $\mathrm{st}_{\mathrm{s}}(x')$ contradicting the assumption that x is a spindle peak.

6. Carathéodory's theorem for spindle convex hull revisited

Recall that Carathéodory's theorem states that the convex hull of a set $X \subset \mathbb{E}^d$ is the union of simplices with vertices in X. We prove the following spindle convex analogue.

Lemma 6.1. *Let X be a set in $\mathbb{E}^d$. If $y \in \operatorname{bd}(\operatorname{conv}_{\mathrm{s}}(X))$, then there exists a subset $\{x_1, \ldots, x_d\}$ of X such that $y \in \operatorname{conv}_{\mathrm{s}}(\{x_1, \ldots, x_d\})$. Moreover, if $y \in \operatorname{int}(\operatorname{conv}_{\mathrm{s}}(X))$, then there exists a set $\{x_1, \ldots, x_{d+1}\} \subseteq X$ such that $y \in \operatorname{int}(\operatorname{conv}_{\mathrm{s}}(\{x_1, \ldots, x_{d+1}\}))$.*

In [1] (see Theorem 5.7) this statement is proved in the special case when X is closed. Since we need it in the general case, we outline the proof here.

Sketch of the proof of Lemma 6.1. If $y \in \operatorname{int}(\operatorname{conv}_{\mathrm{s}}(X))$, then the claim easily follows from the case when X is closed, by a standard approximation argument. Now, assume that $y \in \operatorname{bd}(\operatorname{conv}_{\mathrm{s}}(X))$. Clearly, $\operatorname{cr} X \leq 1$, as otherwise $\operatorname{conv}_{\mathrm{s}}(X) = \mathbb{E}^d$. By Lemma 3.1 of [1], there is a point p such that $\mathbf{B}[p] \supseteq \operatorname{conv}_{\mathrm{s}}(X)$ and $y \in \mathbb{S}(p)$. There are two cases.

First, assume that $X \cap \mathbb{S}(p)$ is contained in an open hemisphere C of $\mathbb{S}(p)$. Then clearly, $\operatorname{conv}_{\mathrm{s}}(X) \cap \mathbb{S}(p)$ is also contained in C (otherwise $C \cup \mathbf{B}(p)$ would be a spindle convex set containing X and not containing y), and thus, so is y. Then y must be in $\mathrm{Sconv} X \cap \mathbb{S}(p)$ and Carathéodory's theorem for the sphere yields the desired result. Next, assume that $X \cap \mathbb{S}(p)$ is not contained in any open hemisphere of $\mathbb{S}(p)$. Then there are $d + 1$ points in $X \cap \mathbb{S}(p)$ such that the convex hull of those $d+1$ points contains p. Clearly, for some d of those points, we either have that their convex hull still contains p or their spherical convex hull within $\mathbb{S}(p)$ contains y. In both cases, the statement follows. $\square$

The following statement shows that the spindle convex hull may be built "from bottom up" in the same way as the convex hull. On the other hand, one can regard that statement as an extension of Lemma 6.1.

Lemma 6.2. *Let* $X_1, \ldots, X_n$ *be spindle convex sets in* $\mathbb{E}^d, d \geq 2$ *and let* $m = \min\{n, d+1\}$. *Then*

$$\begin{aligned}&\operatorname{conv_s}(X_1 \cup \cdots \cup X_n)\\ &\quad= \bigcup_{1 \leq i_1 < \cdots < i_m \leq n} \left(\bigcup_{x_{i_1} \in X_{i_1}, \ldots, x_{i_m} \in X_{i_m}} \operatorname{conv_s}(\{x_{i_1}, \ldots, x_{i_m}\}) \right).\end{aligned}$$

Proof of Lemma 6.2. First, we prove the following special case.

Sublemma 6.1. *Let* Y *be a spindle convex set in* $\mathbb{E}^d$ *and* $z \in \mathbb{E}^d$. *Then*

$$\operatorname{conv_s}(Y \cup \{z\}) = \bigcup_{y \in Y} [y, z]_{\mathrm{s}}.$$

For the proof of Sublemma 6.1, we quote Lemma 5.6 of [1].

Lemma 6.3. *Let* $A \subset \mathbb{E}^d$ *be a set with* $\operatorname{cr}(A) < 1$, *and let* $\mathbf{B}[q]$ *be a closed unit ball containing* A. *Then*

(i) $A \cap \mathbb{S}(q)$ *is contained in an open hemisphere of* $\mathbb{S}(q)$ *and*
(ii) $\operatorname{conv_s}(A) \cap \mathbb{S}(q) = \operatorname{Sconv}(A \cap \mathbb{S}(q))$.

Proof of Sublemma 6.1. We need to show that the right hand side is spindle convex, for which it is sufficient to show the claim in the special case when Y is the spindle of two points, say $Y = [y_1, y_2]_{\mathrm{s}}$.

First, assume that $\operatorname{cr}(\{y_1, y_2, z\}) < 1$. Denote by $Y' = \bigcup_{y \in Y} [y, z]_{\mathrm{s}}$ and by $\hat{Y} := \operatorname{conv_s}(\{y_1, y_2, z\})$. Clearly, $Y' \subseteq \hat{Y}$. We prove the reverse containment, for which it is sufficient to show that $\operatorname{bd} \hat{Y} \subseteq Y'$.

Let w be a point of $\operatorname{bd} \hat{Y}$ distinct from y_1, y_2 and z. By Corollary 3.4. of [1], there is a supporting unit sphere $\mathbb{S}(q)$ of $\hat{Y}$ through w. By Lemma 6.3, $\mathbb{S}(q) \cap \hat{Y} = \operatorname{Sconv}(\{y_1, y_2, z\} \cap \mathbb{S}(q))$. On the other hand, clearly, $\operatorname{Sconv}(\{y_1, y_2, z\} \cap \mathbb{S}(q)) \subseteq Y'$. Hence, $w \in Y'$ finishing the proof in the case when a ball of radius less than one contains $Y \cup \{z\}$.

Assume now that $\operatorname{cr}(\{y_1, y_2, z\}) = 1$. We may assume that $\mathbf{B}[o]$ is the only unit ball that contains $\{y_1, y_2, z\}$. Clearly, y_1, y_2 and z lie on a great circle of $\mathbb{S}(o)$. If $y_1 = -y_2$, we are done. Otherwise, the shorter great circular arc on $\mathbb{S}(o)$ connecting y_1 and y_2 contains $-z$ (or else, y_1, y_2 and z would be on an open great semi-circle contradicting the assumption that $\operatorname{cr}(\{y_1, y_2, z\}) = 1$).

On the other hand, this great circular arc is contained in $[y_1, y_2]_{\mathrm{s}} \cap \mathbb{S}(o)$. Thus, the right hand side contains $[z, -z]_{\mathrm{s}} = \mathbf{B}[o]$.

Finally, assume that $\mathrm{cr}(\{y_1, y_2, z\}) = \lambda > 1$. Then by the previous paragraph,

$$\mathrm{conv}_{\mathrm{s}}^{\lambda}(Y \cup \{z\}) = \bigcup_{y \in Y} [y, z]_{\mathrm{s}}^{\lambda}$$

both being a ball of radius λ. Clearly, replacing λ by 1 we obtain a larger set on both sides, which may only be $\mathbb{E}^d$. □

Before continuing with the proof of Lemma 6.2 we derive the following spindle starshaped analogue of a theorem of Smith [15] on kernels of starshaped sets. In what follows, *maximality* of a set is taken with respect to containment.

Corollary 6.1. *Let S be a spindle starshaped set in $\mathbb{E}^d$. Then*

$$\ker_{\mathrm{s}}(S) = \cap\{Y \subset S \ :\ Y \textit{ is a maximal spindle convex subset of } S\}.$$

Proof. Since singletons are spindle convex sets and, by Zorn's lemma, every spindle convex subset of S is contained in a maximal spindle convex subset of S, the left hand side contains the right one. The reverse containment follows from Sublemma 6.1. □

Now, we prove Lemma 6.2 for $n = 2$.

Sublemma 6.2. *Let Y, Z be spindle convex sets in $\mathbb{E}^d$. Then*

$$\mathrm{conv}_{\mathrm{s}}(Y \cup Z) = \bigcup_{y \in Y, z \in Z} [y, z]_{\mathrm{s}}.$$

Proof of Sublemma 6.2. One can follow the setup of the proof of Sublemma 6.1 and derive the desired claim from the analogous spherical convexity claim in spherical 3-space, which in fact, follows from the analogous convexity claim in Euclidean 3-space. The relevant somewhat laborious, but straightforward details we leave to the reader. □

Finally, we are ready to prove Lemma 6.2 by induction on n. The details are as follows. Recall that Sublemma 6.2 proves Lemma 6.2 for $n = 2$. Thus, we can assume that Lemma 6.2 holds for all n at most k with $k \geq 2$, and then we prove that it holds for $n = k + 1$ as well. Indeed, if $m = d + 1$, then Lemma 6.2 simply follows from Lemma 6.1 in a straightforward way. So, we may assume that $m = n = k + 1 < d + 1$. Clearly, by induction

$$\mathrm{conv}_{\mathrm{s}}(X_1 \cup \cdots \cup X_k) = \bigcup_{x_1 \in X_1, \ldots, x_k \in X_k} \mathrm{conv}_{\mathrm{s}}(\{x_1, \ldots, x_k\}).$$

Therefore it follows from Sublemma 6.2 in a straightforward way that

$$\begin{aligned}\operatorname{conv_s}(X_1 \cup \cdots \cup X_k \cup X_{k+1}) &= \operatorname{conv_s}(((\operatorname{conv_s}(X_1 \cup \cdots \cup X_k)) \cup X_{k+1}) \\ &= \operatorname{conv_s}\left(\left(\bigcup_{x_1 \in X_1, \ldots, x_k \in X_k} \operatorname{conv_s}(\{x_1, \ldots, x_k\})\right) \cup X_{k+1}\right) \\ &= \bigcup_{x_1 \in X_1, \ldots, x_k \in X_k, x_{k+1} \in X_{k+1}} \operatorname{conv_s}(\{x_1, \ldots, x_k, x_{k+1}\}),\end{aligned}$$

finishing the proof of Lemma 6.2. □

7. Klee-type result: proof of Theorem 1.5

Some of the ideas of the proof come from [3]. The details are as follows.

We recall the notion of *Minkowski difference* of two sets A and B in $\mathbb{E}^d$ (cf. p.133 of [14]):

$$A \sim B = \{x \in \mathbb{E}^d \ : \ x + B \subseteq A\}.$$

Let $S_* = \bigcap_{i \in I} S_i$. By Corollary 1.2, it is sufficient to show that for any $x_1, \ldots, x_{d+1}$ in S_*, we have a translate of K in $\cap_{t=1}^{d+1} \operatorname{st_s}(x_t, S_*)$.

For each $i \in I$, let

$$T_i = \{x \in S_i \ : \ [x, x_t]_s \subseteq S_i \ \ \forall 1 \leq t \leq d+1\}.$$

Denote by $T_* = \bigcap_{i \in I} T_i$. We need to prove that $T_* \sim K \neq \emptyset$. Clearly,

$$T_* \sim K = \bigcap_{i \in I} (T_i \sim K). \tag{6}$$

Hence, by the topological Helly theorem (see for example, [3] and [7]), it suffices to prove that for any choice $i_1, \ldots, i_{d+1}$ of (not necessarily distinct) indices from I,

$$T^0 := \left(\bigcap_{l=1}^{d+1} T_{i_l}\right) \sim K = \bigcap_{l=1}^{d+1} (T_{i_l} \sim K) \tag{7}$$

is spindle starshaped.

Let $S^0 = (\cap_{l=1}^{d+1} S_{i_l}) \sim K = \cap_{l=1}^{d+1} (S_{i_l} \sim K)$. By the hypothesis, S^0 is a non-empty, compact, spindle starshaped set. To finish the proof, we will show that $\ker_s S^0 \subseteq \ker_s T^0$.

Let x be an arbitrary point of $\ker_s S^0$ and $y \in T^0$. We need to show that $[x, y]_s \subseteq T^0$. We fix an index i_l, say i_1. We need $[x, y]_s \subseteq T_{i_1} \sim K$ or, equivalently, that for any $u \in [x, y]_s + K$ we have $[u, x_t]_s \subseteq S_{i_1}$ for any $t = 1, \ldots, d+1$.

On the one hand, since $y \in T^0$, we have that

$$C_t := \operatorname{conv_s}\left(((y + K) \cup \{x_t\})\right) \subseteq \bigcap_{l=1}^{d+1} S_{i_l} \tag{8}$$

holds for all $1 \leq t \leq d+1$. On the other hand, $x + K \subseteq \ker_s\left(\cap_{l=1}^{d+1} S_{i_l}\right)$.

It follows that

$$\cup\{[a,b]_{\mathrm{s}} \,:\, a \in x+K, b \in C_t\} \subseteq \bigcap_{l=1}^{d+1} S_{i_l}$$

holds for all $1 \leq t \leq d+1$. We may assume that K is spindle convex, since the spindle kernel of a spindle starshaped set is spindle convex. Thus, Lemma 6.2 (more exactly, Sublemma 6.2) implies that

$$\cup\left\{[a,b]_{\mathrm{s}} \,:\, a \in x+K, b \in C_t\right\} = \operatorname{conv}_{\mathrm{s}}\left(((x+K) \cup C_t)\right), \tag{9}$$

which clearly contains $[u, x_t]_{\mathrm{s}}$, finishing the proof of Theorem 1.5.

8. Countable intersections: proof of Theorem 1.6

First, we prove the following: Let $\mathcal{F}$ be a family of spindle starshaped sets in $\mathbb{E}^d$ with the property that the intersection of any countable subfamily of $\mathcal{F}$ is a spindle starshaped (hence, non-empty) set. Then $\bigcap \mathcal{F}$ is also starshaped. In fact, this is the special case of Theorem 1.6 where K is a singleton. The details are as follows.

First, we define a new family, $\mathcal{F}_1$ as follows. We enumerate members of $\mathbb{Q}^d$ and carry out the following inductive algorithm. For each $q \in \mathbb{Q}^d$ if there is an F in our set family such that $q \notin F$ then we intersect each member of the set family by F to obtain the next set family. At the end of this algorithm, we obtain the family $\mathcal{F}_1$ whose countable subfamilies clearly have spindle starshaped intersections.

Now, $\mathbb{Q}^d \cap F$ is the same set for all $F \in \mathcal{F}_1$. If this set is empty or a singleton, then each set in $\mathcal{F}$ is a singleton (otherwise they would have a non-empty interior and hence, contain a rational point). Clearly, these singletons must be identical, and the theorem follows.

So we assume that $\mathbb{Q}^d \cap F$ contains more than one point (for each $F \in \mathcal{F}$). For a set $A \subset \mathbb{E}^d$ we define its rational spindle kernel as

$$\ker_{\mathrm{s}}^{\mathbb{Q}} A = \left\{p \in A \,:\, [p,a]_{\mathrm{s}} \cap \mathbb{Q}^d \subseteq A \text{ for each } a \in A \cap \mathbb{Q}^d\right\}.$$

Note that $\ker_{\mathrm{s}}^{\mathbb{Q}} A$ may contain non-rational points.

By definition and the fact that $\mathbb{Q}^d \cap F$ is the same set for all $F \in \mathcal{F}_1$, the following hold:

$$\ker_{\mathrm{s}}^{\mathbb{Q}} F \supseteq \ker_{\mathrm{s}} F \neq \emptyset \quad \text{ for any } F \in \mathcal{F}_1, \tag{10}$$

and

$$\ker_{\mathrm{s}}^{\mathbb{Q}}(F_1 \cap F_2) = \ker_{\mathrm{s}}^{\mathbb{Q}} F_1 \cap \ker_{\mathrm{s}}^{\mathbb{Q}} F_2 \quad \text{ for any } F_1, F_2 \in \mathcal{F}_1. \tag{11}$$

Assume that $\ker_{\mathrm{s}}^{\mathbb{Q}} F_1$ is a singleton, say $\{p\}$ for some $F_1 \in \mathcal{F}_1$. Then by (10) and (11), p is in $\ker_{\mathrm{s}} \bigcap \mathcal{F}_1$.

Thus, we may assume that for all $F \in \mathcal{F}_1$, its rational spindle kernel $\ker_{\mathrm{s}}^{\mathbb{Q}} F$ is not a singleton. Using Lemma 6.2, we have that

$$\text{if } p, q \in \ker_{\mathrm{s}}^{\mathbb{Q}} F \text{ then } \operatorname{int}[p,q]_{\mathrm{s}} \cap \mathbb{Q}^d \subseteq \left(\ker_{\mathrm{s}}^{\mathbb{Q}} F\right) \cap \mathbb{Q}^d. \tag{12}$$

Note that $(\ker_{\mathrm{s}}^{\mathbb{Q}} F) \cap \mathbb{Q}^d$ is the same set for all $F \in \mathcal{F}_1$. It follows from (12) that the interior of $(\ker_{\mathrm{s}}^{\mathbb{Q}} F) \cap \mathbb{Q}^d$ relative to $\mathbb{Q}^d$ is not empty. Using the fact that F is spindle starshaped, it is not difficult to see that for any interior (relative to $\mathbb{Q}^d$) point p of $(\ker_{\mathrm{s}}^{\mathbb{Q}} F) \cap \mathbb{Q}^d$, we have that $p \in \ker_{\mathrm{s}} F$. Thus for any such p, we have $p \in \ker_{\mathrm{s}} (\cap \mathcal{F}_1) = \ker_{\mathrm{s}} (\cap \mathcal{F})$ finishing the proof.

To prove the general case (ie., when K is not a singleton), we may follow the above proof up to (10), from which

$$\left(\ker_{\mathrm{s}}^{\mathbb{Q}} F\right) \sim K \supseteq (\ker_{\mathrm{s}} F) \sim K \neq \emptyset \quad \text{for any } F \in \mathcal{F}_1,$$

follows. Similarly, from (11) we obtain

$$\left(\ker_{\mathrm{s}}^{\mathbb{Q}}(F_1 \cap F_2)\right) \sim K = \left[\left(\ker_{\mathrm{s}}^{\mathbb{Q}} F_1\right) \sim K\right] \cap \left[\left(\ker_{\mathrm{s}}^{\mathbb{Q}} F_2\right) \sim K\right]$$

holds for any $F_1, F_2 \in \mathcal{F}_1$. Again, we may assume that for all $F \in \mathcal{F}_1$, $\ker_{\mathrm{s}}^{\mathbb{Q}}(F) \sim K$ is not a singleton, otherwise the theorem follows easily. From (12) we obtain

$$\text{if } p, q \in (\ker_{\mathrm{s}}^{\mathbb{Q}} F) \sim K \text{ then } \operatorname{int}[p,q]_{\mathrm{s}} \cap \mathbb{Q}^d \subseteq \left((\ker_{\mathrm{s}}^{\mathbb{Q}} F) \sim K\right) \cap \mathbb{Q}^d.$$

Now, by the same inductive procedure that we used at the beginning of the proof above, we may assume that $((\ker_{\mathrm{s}}^{\mathbb{Q}} F) \sim K) \cap \mathbb{Q}^d$ is the same set for all $F \in \mathcal{F}_1$. Finally, for any interior (relative to $\mathbb{Q}^d$) point p of $((\ker_{\mathrm{s}}^{\mathbb{Q}} F) \sim K) \cap \mathbb{Q}^d$, we have that $p \in (\ker_{\mathrm{s}} \cap \mathcal{F}) \sim K$.

9. Art gallery: proofs of Theorems 1.7 and 1.8

Remark 9.1. The assumption of simple connectedness cannot be dropped. This is shown by the example of an annulus. For any $N \in \mathbb{Z}^+$, if the inner circle is small enough then any N points of the outer circle can be seen from some point of the annulus, but no point sees all the points (or a sufficiently large finite subset) of the outer circle.

This example can be turned into one in three–space: for any $N \in \mathbb{Z}^+$ there is a homology cell S in $\mathbb{R}^3$ such that any N points of a certain subset of S can be seen from some point of S but no point sees them all. We leave it as an exercise to the reader.

We note that Remark 9.1 applies to this spindle version of the theorem as well.

Proof of Theorem 1.7. We call a set F in S *geodesically convex* with respect to S if for any $p, q \in F$ the shortest path in S connecting p and q (which is unique by the simple connectedness of S) is contained in F. We claim the following:

1. The intersection of geodesically convex sets is again geodesically convex (w.r.t. S).
2. A geodesically convex set is simply connected.
3. $\mathrm{st}\,(x, S)$ is compact and geodesically convex for any $x \in S$.

1. is obvious. 2. is easy to prove. Indeed, consider a subset F of S that is not simply connected. Then there is a line ℓ through some point of $F \setminus S$ whose intersection with F is not connected. It clearly shows that F is not geodesically convex.

To prove 3., let p and q be points of $\mathrm{st}\,(x, S)$, and consider the shortest path γ connecting them within S. If x, p and q are collinear, then γ is simply the line segment $[p, q]$, which is in $\mathrm{st}\,(x, S)$. If they are not collinear then the rays $\overrightarrow{xp}, \overrightarrow{xq}$ bound two angular regions on the plane, one of which is convex, call it T. We may assume that γ is not $[p, x] \cup [x, q]$, as otherwise we are done. Let p' (resp. q') be the point of $\gamma \cap \overrightarrow{xp}$ (resp. $\gamma \cap \overrightarrow{xq}$) closest to x. Consider the part γ' of γ from p' to q'. Clearly, $\gamma = [p, p'] \cup \gamma' \cup [q', q]$. Now, neither p' nor q' is x. Moreover, clearly, $\gamma' \subset T$. By the simple connectedness of S, we have that $\gamma' \subset \mathrm{st}\,(x, S)$ finishing the proof of the geodesic convexity of $\mathrm{st}\,(x, S)$. Its compactness follows from the compactness of S.

Finally, Theorem 1.7 follows from these claims and the topological version of Helly's theorem. ☐

Proof of Theorem 1.8. The only point where the proof of Theorem 1.7 needs to be changed a bit is the proof of 3. First, we notice that by the simple connectedness of S, γ' lies in the triangle $\Delta = \Delta_{p'xq'}$. It is not difficult to see that for any point y of this triangle, $[x, y]_{\mathrm{s}} \subseteq \Delta \cup [x, p']_{\mathrm{s}} \cup [x, q']_{\mathrm{s}}$. Now, again, the simple connectedness of S yields that each point of γ' is spindle visible from x. ☐

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