

# Minimizing the mean projections of finite $\rho$ -separable packings

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**Abstract** A packing of translates of a convex body in the  $d$ -dimensional Euclidean space  $\mathbb{E}^d$  is said to be totally separable if any two packing elements can be separated by a hyperplane of  $\mathbb{E}^d$  disjoint from the interior of every packing element. We call the packing  $\mathcal{P}$  of translates of a centrally symmetric convex body  $\mathbf{C}$  in  $\mathbb{E}^d$  a  $\rho$ -separable packing for given  $\rho \geq 1$  if in every ball concentric to a packing element of  $\mathcal{P}$  having radius  $\rho$  (measured in the norm generated by  $\mathbf{C}$ ) the corresponding sub-packing of  $\mathcal{P}$  is totally separable. The main result of this paper is the following theorem. Consider the convex hull  $\mathbf{Q}$  of  $n$  non-overlapping translates of an arbitrary centrally symmetric convex body  $\mathbf{C}$  forming a  $\rho$ -separable packing in  $\mathbb{E}^d$  with  $n$  being sufficiently large for given  $\rho \geq 1$ . If  $\mathbf{Q}$  has minimal mean  $i$ -dimensional projection for given  $i$  with  $1 \leq i < d$ , then  $\mathbf{Q}$  is approximately a  $d$ -dimensional ball. This extends a theorem of Böröczky (Monatsh Math 118:41–54, 1994) from translative packings to  $\rho$ -separable translative packings for  $\rho \geq 1$ .

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### 1 Introduction

We denote the  $d$ -dimensional Euclidean space by  $\mathbb{E}^d$ . Let  $\mathbf{B}^d$  denote the unit ball centered at the origin  $\mathbf{o}$  in  $\mathbb{E}^d$ . A  $d$ -dimensional convex body  $\mathbf{C}$  is a compact convex subset of  $\mathbb{E}^d$  with non-empty interior  $\text{int } \mathbf{C}$ . (If  $d = 2$ , then  $\mathbf{C}$  is said to be a *convex domain*.) If  $\mathbf{C} = -\mathbf{C}$ , where  $-\mathbf{C} = \{-x : x \in \mathbf{C}\}$ , then  $\mathbf{C}$  is said to be  *$\mathbf{o}$ -symmetric* and a translate  $\mathbf{c} + \mathbf{C}$  of  $\mathbf{C}$  is called centrally symmetric with center  $\mathbf{c}$ .

The starting point as well as the main motivation for writing this paper is the following elegant theorem of Böröczky [8]: consider the convex hull  $\mathbf{Q}$  of  $n$  non-overlapping translates of an arbitrary convex body  $\mathbf{C}$  in  $\mathbb{E}^d$  with  $n$  being sufficiently large. If  $\mathbf{Q}$  has minimal mean  $i$ -dimensional projection for given  $i$  with  $1 \leq i < d$ , then  $\mathbf{Q}$  is approximately a  $d$ -dimensional ball. In this paper, our main goal is to prove an extension of this theorem to  $\rho$ -separable translative packings of convex bodies in  $\mathbb{E}^d$ . Next, we define the concept of  $\rho$ -separable translative packings and then state our main result.

A packing of translates of a convex domain  $\mathbf{C}$  in  $\mathbb{E}^2$  is said to be *totally separable* if any two packing elements can be separated by a line of  $\mathbb{E}^2$  disjoint from the interior of every packing element. This notion was introduced by Fejes Tóth and Fejes Tóth [9]. We can define a totally separable packing of translates of a  $d$ -dimensional convex body  $\mathbf{C}$  in a similar way by requiring any two packing elements to be separated by a hyperplane in  $\mathbb{E}^d$  disjoint from the interior of every packing element [6, 7].

**Definition 1** Let  $\mathbf{C}$  be an  $\mathbf{o}$ -symmetric convex body of  $\mathbb{E}^d$ . Furthermore, let  $\|\cdot\|_{\mathbf{C}}$  denote the norm generated by  $\mathbf{C}$ , i.e., let  $\|\mathbf{x}\|_{\mathbf{C}} := \inf\{\lambda \mid \mathbf{x} \in \lambda\mathbf{C}\}$  for any  $\mathbf{x} \in \mathbb{E}^d$ . Now, let  $\rho \geq 1$ . We say that the packing

$$\mathcal{P}_{\text{sep}} := \{\mathbf{c}_i + \mathbf{C} \mid i \in I \text{ with } \|\mathbf{c}_j - \mathbf{c}_k\|_{\mathbf{C}} \geq 2 \text{ for all } j \neq k \in I\}$$

of (finitely or infinitely many) non-overlapping translates of  $\mathbf{C}$  with centers  $\{\mathbf{c}_i \mid i \in I\}$  is a  $\rho$ -separable packing in  $\mathbb{E}^d$  if for each  $i \in I$  the finite packing  $\{\mathbf{c}_j + \mathbf{C} \mid \mathbf{c}_j + \mathbf{C} \subseteq \mathbf{c}_i + \rho\mathbf{C}\}$  is a totally separable packing (in  $\mathbf{c}_i + \rho\mathbf{C}$ ). Finally, let  $\delta_{\text{sep}}(\rho, \mathbf{C})$  denote the largest density of all  $\rho$ -separable translative packings of  $\mathbf{C}$  in  $\mathbb{E}^d$ , i.e., let

$$\delta_{\text{sep}}(\rho, \mathbf{C}) := \sup_{\mathcal{P}_{\text{sep}}} \left( \limsup_{\lambda \rightarrow +\infty} \frac{\sum_{\mathbf{c}_i + \mathbf{C} \subseteq \mathbf{W}_\lambda^d} \text{vol}_d(\mathbf{c}_i + \mathbf{C})}{\text{vol}_d(\mathbf{W}_\lambda^d)} \right),$$

where  $\mathbf{W}_\lambda^d$  denotes the  $d$ -dimensional cube of edge length  $2\lambda$  centered at  $\mathbf{o}$  in  $\mathbb{E}^d$  having edges parallel to the coordinate axes of  $\mathbb{E}^d$  and  $\text{vol}_d(\cdot)$  refers to the  $d$ -dimensional volume of the corresponding set in  $\mathbb{E}^d$ .

*Remark 1* Let  $\delta(\mathbf{C})$  (resp.,  $\delta_{\text{sep}}(\mathbf{C})$ ) denote the supremum of the upper densities of all translative packings (resp., totally separable translative packings) of the  $\mathbf{o}$ -symmetric convex body  $\mathbf{C}$  in  $\mathbb{E}^d$ . Clearly,  $\delta_{\text{sep}}(\mathbf{C}) \leq \delta_{\text{sep}}(\rho, \mathbf{C}) \leq \delta(\mathbf{C})$  for all  $\rho \geq 1$ . Furthermore, if  $1 \leq \rho < 3$ , then any  $\rho$ -separable translative packing of  $\mathbf{C}$  in  $\mathbb{E}^d$  is simply a translative packing of  $\mathbf{C}$  and therefore,  $\delta_{\text{sep}}(\rho, \mathbf{C}) = \delta(\mathbf{C})$ .

Recall that the mean  $i$ -dimensional projection  $M_i(\mathbf{C})$  ( $i = 1, 2, \dots, d - 1$ ) of the convex body  $\mathbf{C}$  in  $\mathbb{E}^d$ , can be expressed ([13]) with the help of mixed volume via the formula

$$M_i(\mathbf{C}) = \frac{\kappa_i}{\kappa_d} V(\overbrace{\mathbf{C}, \dots, \mathbf{C}}^i, \overbrace{\mathbf{B}^d, \dots, \mathbf{B}^d}^{d-i}),$$

where  $\kappa_d$  is the volume of  $\mathbf{B}^d$  in  $\mathbb{E}^d$ . Note that  $M_i(\mathbf{B}^d) = \kappa_i$ , and the surface area of  $\mathbf{C}$  is  $S(\mathbf{C}) = \frac{d\kappa_d}{\kappa_{d-1}} M_{d-1}(\mathbf{C})$  and in particular,  $S(\mathbf{B}^d) = d\kappa_d$ . Set  $M_d(\mathbf{C}) := \text{vol}_d(\mathbf{C})$ . Finally, let  $R(\mathbf{C})$  (resp.,  $r(\mathbf{C})$ ) denote the circumradius (resp., inradius) of the convex body  $\mathbf{C}$  in  $\mathbb{E}^d$ , which is the radius of the smallest (resp., largest) ball that contains (resp., is contained in)  $\mathbf{C}$ . Our main result is the following.

**Theorem 1** *Let  $d \geq 2$ ,  $1 \leq i \leq d - 1$ ,  $\rho \geq 1$ , and let  $\mathbf{Q}$  be the convex hull of the  $\rho$ -separable packing of  $n$  translates of the  $\mathbf{o}$ -symmetric convex body  $\mathbf{C}$  in  $\mathbb{E}^d$  such that  $M_i(\mathbf{Q})$  is minimal and  $n \geq \frac{4^d d^{4d}}{\delta_{\text{sep}}(\rho, \mathbf{C})^{d-1}} \cdot \left(\rho \frac{R(\mathbf{C})}{r(\mathbf{C})}\right)^d$ . Then*

$$\frac{r(\mathbf{Q})}{R(\mathbf{Q})} \geq 1 - \frac{\omega}{n^{\frac{2}{d(d+3)}}}, \tag{1}$$

for  $\omega = \lambda(d) \left(\frac{\rho R(\mathbf{C})}{r(\mathbf{C})}\right)^{\frac{2}{d+3}}$ , where  $\lambda(d)$  depends only on the dimension  $d$ . In addition,

$$M_i(\mathbf{Q}) = \left(1 + \frac{\sigma}{n^{\frac{1}{d}}}\right) M_i(\mathbf{B}^d) \left(\frac{\text{vol}_d(\mathbf{C})}{\delta_{\text{sep}}(\rho, \mathbf{C})\kappa_d}\right)^{\frac{i}{d}} \cdot n^{\frac{i}{d}},$$

where  $-\frac{2.25R(\mathbf{C})\rho di}{r(\mathbf{C})\delta_{\text{sep}}(\rho, \mathbf{C})} \leq \sigma \leq \frac{2.1R(\mathbf{C})\rho i}{r(\mathbf{C})\delta_{\text{sep}}(\rho, \mathbf{C})}$ .

*Remark 2* It is worth restating Theorem 1 as follows: Consider the convex hull  $\mathbf{Q}$  of  $n$  non-overlapping translates of an arbitrary  $\mathbf{o}$ -symmetric convex body  $\mathbf{C}$  forming a  $\rho$ -separable packing in  $\mathbb{E}^d$  with  $n$  being sufficiently large. If  $\mathbf{Q}$  has minimal mean  $i$ -dimensional projection for given  $i$  with  $1 \leq i < d$ , then  $\mathbf{Q}$  is approximately a  $d$ -dimensional ball.

*Remark 3* The nature of the analogue question on minimizing  $M_d(\mathbf{Q}) = \text{vol}_d(\mathbf{Q})$  is very different. Namely, recall that Betke and Henk [4] proved L. Fejes Tóth’s sausage conjecture for  $d \geq 42$  according to which the smallest volume of the convex hull of  $n$  non-overlapping unit balls in  $\mathbb{E}^d$  is obtained when the  $n$  unit balls form a sausage, that is, a linear packing (see also [2] and [3]). As linear packings of unit balls are

$\rho$ -separable therefore the above theorem of Betke and Henk applies to  $\rho$ -separable packings of unit balls in  $\mathbb{E}^d$  for all  $\rho \geq 1$  and  $d \geq 42$ . On the other hand, the problem of minimizing the volume of the convex hull of  $n$  unit balls forming a  $\rho$ -separable packing in  $\mathbb{E}^d$  remains an interesting open problem for  $\rho \geq 1$  and  $2 \leq d < 42$ . Last but not least, the problem of minimizing  $M_d(\mathbf{Q})$  for  $\mathbf{o}$ -symmetric convex bodies  $\mathbf{C}$  different from a ball in  $\mathbb{E}^d$  seems to be wide open for  $\rho \geq 1$  and  $d \geq 2$ .

*Remark 4* Let  $d \geq 2, 1 \leq i \leq d - 1, n > 1$ , and let  $\mathbf{C}$  be a given  $\mathbf{o}$ -symmetric convex body in  $\mathbb{E}^d$ . Furthermore, let  $\mathbf{Q}$  be the convex hull of the totally separable packing of  $n$  translates of  $\mathbf{C}$  in  $\mathbb{E}^d$  such that  $M_i(\mathbf{Q})$  is minimal. Then it is natural to ask for the limit shape of  $\mathbf{Q}$  as  $n \rightarrow +\infty$ , that is, to ask for an analogue of Theorem 1 within the family of totally separable translative packings of  $\mathbf{C}$  in  $\mathbb{E}^d$ . This would require some new ideas besides the ones used in the following proof of Theorem 1.

In the rest of the paper by adopting the method of Böröczky [8] and making the necessary modifications, we give a proof of Theorem 1.

## 2 Basic properties of finite $\rho$ -separable translative packings

The following statement is the  $\rho$ -separable analogue of the Lemma in [5] (see also Theorem 3.1 in [2]).

**Lemma 1** *Let  $\{\mathbf{c}_i + \mathbf{C} \mid 1 \leq i \leq n\}$  be an arbitrary  $\rho$ -separable packing of  $n$  translates of the  $\mathbf{o}$ -symmetric convex body  $\mathbf{C}$  in  $\mathbb{E}^d$  with  $\rho \geq 1, n \geq 1$ , and  $d \geq 2$ . Then*

$$\frac{n \text{vol}_d(\mathbf{C})}{\text{vol}_d(\cup_{i=1}^n \mathbf{c}_i + 2\rho\mathbf{C})} \leq \delta_{\text{sep}}(\rho, \mathbf{C}).$$

*Proof* We use the method of the proof of the Lemma in [5] (resp., Theorem 3.1 in [2]) with proper modifications. The details are as follows. Assume that the claim is not true. Then there is an  $\epsilon > 0$  such that

$$\text{vol}_d(\cup_{i=1}^n \mathbf{c}_i + 2\rho\mathbf{C}) = \frac{n \text{vol}_d(\mathbf{C})}{\delta_{\text{sep}}(\rho, \mathbf{C})} - \epsilon \tag{2}$$

Let  $C_n = \{\mathbf{c}_i \mid i = 1, \dots, n\}$  and let  $\Lambda$  be a packing lattice of  $C_n + 2\rho\mathbf{C} = \cup_{i=1}^n \mathbf{c}_i + 2\rho\mathbf{C}$  such that  $C_n + 2\rho\mathbf{C}$  is contained in a fundamental parallelotope of  $\Lambda$  say, in  $\mathbf{P}$ , which is symmetric about the origin. Recall that for each  $\lambda > 0, \mathbf{W}_\lambda^d$  denotes the  $d$ -dimensional cube of edge length  $2\lambda$  centered at the origin  $\mathbf{o}$  in  $\mathbb{E}^d$  having edges parallel to the coordinate axes of  $\mathbb{E}^d$ . Clearly, there is a constant  $\mu > 0$  depending on  $\mathbf{P}$  only, such that for each  $\lambda > 0$  there is a subset  $L_\lambda$  of  $\Lambda$  with

$$\mathbf{W}_\lambda^d \subseteq L_\lambda + \mathbf{P} \text{ and } L_\lambda + 2\mathbf{P} \subseteq \mathbf{W}_{\lambda+\mu}^d. \tag{3}$$

The definition of  $\delta_{\text{sep}}(\rho, \mathbf{C})$  implies that for each  $\lambda > 0$  there exists a  $\rho$ -separable packing of  $m(\lambda)$  translates of  $\mathbf{C}$  in  $\mathbb{E}^d$  with centers at the points of  $C(\lambda)$  such that

$$C(\lambda) + \mathbf{C} \subset \mathbf{W}_\lambda^d$$

and

$$\lim_{\lambda \rightarrow +\infty} \frac{m(\lambda)\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{W}_\lambda^d)} = \delta_{\text{sep}}(\rho, \mathbf{C}).$$

As  $\lim_{\lambda \rightarrow +\infty} \frac{\text{vol}_d(\mathbf{W}_{\lambda+\mu}^d)}{\text{vol}_d(\mathbf{W}_\lambda^d)} = 1$  therefore there exist  $\xi > 0$  and a  $\rho$ -separable packing of  $m(\xi)$  translates of  $\mathbf{C}$  in  $\mathbb{E}^d$  with centers at the points of  $C(\xi)$  and with  $C(\xi) + \mathbf{C} \subset \mathbf{W}_\xi^d$  such that

$$\frac{\text{vol}_d(\mathbf{P})\delta_{\text{sep}}(\rho, \mathbf{C})}{\text{vol}_d(\mathbf{P}) + \epsilon} < \frac{m(\xi)\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{W}_{\xi+\mu}^d)} \text{ and } \frac{n\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{P}) + \epsilon} < \frac{n\text{vol}_d(\mathbf{C})\text{card}(L_\xi)}{\text{vol}_d(\mathbf{W}_{\xi+\mu}^d)}, \tag{4}$$

where  $\text{card}(\cdot)$  refers to the cardinality of the given set. Now, for each  $\mathbf{x} \in \mathbf{P}$  we define a  $\rho$ -separable packing of  $\bar{m}(\mathbf{x})$  translates of  $\mathbf{C}$  in  $\mathbb{E}^d$  with centers at the points of

$$\bar{C}(\mathbf{x}) := \{\mathbf{x} + L_\xi + C_n\} \cup \{\mathbf{y} \in C(\xi) \mid \mathbf{y} \notin \mathbf{x} + L_\xi + C_n + \text{int}(2\rho\mathbf{C})\}.$$

Clearly, (3) implies that  $\bar{C}(\mathbf{x}) + \mathbf{C} \subset \mathbf{W}_{\xi+\mu}^d$ . Now, in order to evaluate  $\int_{\mathbf{x} \in \mathbf{P}} \bar{m}(\mathbf{x})d\mathbf{x}$ , we introduce the function  $\chi_{\mathbf{y}}$  for each  $\mathbf{y} \in C(\xi)$  defined as follows:  $\chi_{\mathbf{y}}(\mathbf{x}) = 1$  if  $\mathbf{y} \notin \mathbf{x} + L_\xi + C_n + \text{int}(2\rho\mathbf{C})$  and  $\chi_{\mathbf{y}}(\mathbf{x}) = 0$  for any other  $\mathbf{x} \in \mathbf{P}$ . Based on the origin symmetric  $\mathbf{P}$  it is easy to see that for any  $\mathbf{y} \in C(\xi)$  one has  $\int_{\mathbf{x} \in \mathbf{P}} \chi_{\mathbf{y}}(\mathbf{x})d\mathbf{x} = \text{vol}_d(\mathbf{P}) - \text{vol}_d(C_n + 2\rho\mathbf{C})$ . Thus, it follows in a straightforward way that

$$\begin{aligned} \int_{\mathbf{x} \in \mathbf{P}} \bar{m}(\mathbf{x})d\mathbf{x} &= \int_{\mathbf{x} \in \mathbf{P}} (n\text{card}(L_\xi) + \sum_{\mathbf{y} \in C(\xi)} \chi_{\mathbf{y}}(\mathbf{x}))d\mathbf{x} \\ &= n\text{vol}_d(\mathbf{P})\text{card}(L_\xi) + m(\xi)(\text{vol}_d(\mathbf{P}) - \text{vol}_d(C_n + 2\rho\mathbf{C})). \end{aligned}$$

Hence, there is a point  $\mathbf{p} \in \mathbf{P}$  with

$$\bar{m}(\mathbf{p}) \geq m(\xi) \left( 1 - \frac{\text{vol}_d(C_n + 2\rho\mathbf{C})}{\text{vol}_d(\mathbf{P})} \right) + n\text{card}(L_\xi)$$

and so

$$\frac{\bar{m}(\mathbf{p})\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{W}_{\xi+\mu}^d)} \geq \frac{m(\xi)\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{W}_{\xi+\mu}^d)} \left( 1 - \frac{\text{vol}_d(C_n + 2\rho\mathbf{C})}{\text{vol}_d(\mathbf{P})} \right) + \frac{n\text{vol}_d(\mathbf{C})\text{card}(L_\xi)}{\text{vol}_d(\mathbf{W}_{\xi+\mu}^d)}. \tag{5}$$

Now, (2) implies in a straightforward way that

$$\frac{\text{vol}_d(\mathbf{P})\delta_{\text{sep}}(\rho, \mathbf{C})}{\text{vol}_d(\mathbf{P}) + \epsilon} \left( 1 - \frac{\text{vol}_d(C_n + 2\rho\mathbf{C})}{\text{vol}_d(\mathbf{P})} \right) + \frac{n\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{P}) + \epsilon} = \delta_{\text{sep}}(\rho, \mathbf{C}) \tag{6}$$

Thus, (4), (5), and (6) yield that

$$\frac{\overline{m}(\mathbf{p})\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{W}_{\xi+\mu}^d)} > \delta_{\text{sep}}(\rho, \mathbf{C}).$$

As  $\overline{C}(\mathbf{p}) + \mathbf{C} \subset \mathbf{W}_{\xi+\mu}^d$  this contradicts the definition of  $\delta_{\text{sep}}(\rho, \mathbf{C})$ , finishing the proof of Lemma 1. □

**Definition 2** Let  $d \geq 2$ ,  $\rho \geq 1$ , and let  $\mathbf{K}$  (resp.,  $\mathbf{C}$ ) be a convex body (resp., an  $\mathbf{o}$ -symmetric convex body) in  $\mathbb{E}^d$ . Then let  $\nu_{\mathbf{C}}(\rho, \mathbf{K})$  denote the largest  $n$  with the property that there exists a  $\rho$ -separable packing  $\{\mathbf{c}_i + \mathbf{C} \mid 1 \leq i \leq n\}$  such that  $\{\mathbf{c}_i \mid 1 \leq i \leq n\} \subset \mathbf{K}$ .

**Lemma 2** Let  $d \geq 2$ ,  $\rho \geq 1$ , and let  $\mathbf{K}$  (resp.,  $\mathbf{C}$ ) be a convex body (resp., an  $\mathbf{o}$ -symmetric convex body) in  $\mathbb{E}^d$ . Then

$$\left(1 + \frac{2\rho R(\mathbf{C})}{r(\mathbf{K})}\right)^{-d} \frac{\text{vol}_d(\mathbf{C})\nu_{\mathbf{C}}(\rho, \mathbf{K})}{\delta_{\text{sep}}(\rho, \mathbf{C})} \leq \text{vol}_d(\mathbf{K}) \leq \frac{\text{vol}_d(\mathbf{C})\nu_{\mathbf{C}}(\rho, \mathbf{K})}{\delta_{\text{sep}}(\rho, \mathbf{C})}.$$

*Proof* Observe that Lemma 1 and the containments  $\mathbf{K} + 2\rho\mathbf{C} \subseteq \left(1 + \frac{2\rho R(\mathbf{C})}{r(\mathbf{K})}\right) \mathbf{K}$  yield the lower bound immediately.

We prove the upper bound. Let  $0 < \varepsilon < \delta_{\text{sep}}(\rho, \mathbf{C})$ . By the definition of  $\delta_{\text{sep}}(\rho, \mathbf{C})$ , if  $\lambda$  is sufficiently large, then there is a  $\rho$ -separable packing  $\{\mathbf{c}_i + \mathbf{C} \mid 1 \leq i \leq n\}$  such that  $C_n := \{\mathbf{c}_i \mid 1 \leq i \leq n\} \subset \mathbf{W}_{\lambda}^d$  and

$$\frac{n\text{vol}_d(\mathbf{C})}{\text{vol}_d(\mathbf{W}_{\lambda}^d)} \geq \delta_{\text{sep}}(\rho, \mathbf{C}) - \varepsilon. \tag{7}$$

**SubLemma 1** If  $\mathbf{X}$  and  $\mathbf{Y}$  are convex bodies in  $\mathbb{E}^d$  and  $\mathbf{C}$  is an  $\mathbf{o}$ -symmetric convex body in  $\mathbb{E}^d$ , then

$$\nu_{\mathbf{C}}(\rho, \mathbf{Y}) \geq \frac{\text{vol}_d(\mathbf{Y})\nu_{\mathbf{C}}(\rho, \mathbf{X})}{\text{vol}_d(\mathbf{X} - \mathbf{Y})}. \tag{8}$$

*Proof* Indeed, consider any finite point set  $X := \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbf{X}$ . Observe that the following are equivalent for a positive integer  $k$ :

- $k$  is the maximum number a point of  $\mathbf{X} - \mathbf{Y}$  is covered by the sets  $\mathbf{x}_i - \mathbf{Y}$ ,  $\mathbf{x}_i \in X$ ,
- $k$  is the maximum number such that  $\text{card}((\mathbf{z} + \mathbf{Y}) \cap X) = k$  for some point  $\mathbf{z} \in \mathbf{X} - \mathbf{Y}$ .

Thus,  $N \text{vol}_d(\mathbf{Y}) \leq \text{card}((\mathbf{z} + \mathbf{Y}) \cap X) \text{vol}_d(\mathbf{X} - \mathbf{Y})$  for some  $\mathbf{z} \in \mathbf{X} - \mathbf{Y}$ . Hence, if  $\{\mathbf{x}_i + \mathbf{C} \mid 1 \leq i \leq N\}$  is an arbitrary  $\rho$ -separable packing with  $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbf{X}$ , then

$$\nu_{\mathbf{C}}(\rho, \mathbf{Y}) \geq \text{card}((\mathbf{z} + \mathbf{Y}) \cap X) \geq \frac{\text{vol}_d(\mathbf{Y})N}{\text{vol}_d(\mathbf{X} - \mathbf{Y})},$$

which implies (8). □

Applying (8) to  $\mathbf{X} = \mathbf{W}_\lambda^d$  and  $\mathbf{Y} = -\mathbf{K}$  and using (7), we obtain

$$v_{\mathbf{C}}(\rho, \mathbf{K}) \geq \frac{n \operatorname{vol}_d(\mathbf{K})}{\operatorname{vol}_d(\mathbf{W}_\lambda^d + \mathbf{K})} \geq \frac{\operatorname{vol}_d(\mathbf{K})}{\operatorname{vol}_d(\mathbf{W}_{\lambda+R(\mathbf{K})}^d)} \cdot \frac{\operatorname{vol}_d(\mathbf{W}_\lambda^d)(\delta_{\text{sep}}(\rho, \mathbf{C}) - \varepsilon)}{\operatorname{vol}_d(\mathbf{C})},$$

which finishes the proof of Lemma 2. □

**Definition 3** Let  $d \geq 2, n \geq 1, \rho \geq 1$ , and let  $\mathbf{C}$  be an  $\mathbf{o}$ -symmetric convex body in  $\mathbb{E}^d$ . Then let  $R_{\mathbf{C}}(\rho, n)$  be the smallest radius  $R > 0$  with the property that  $v_{\mathbf{C}}(\rho, R \mathbf{B}^d) \geq n$ .

Clearly, for any  $\varepsilon > 0$  we have  $v_{\mathbf{C}}(\rho, (R_{\mathbf{C}}(\rho, n) - \varepsilon) \mathbf{B}^d) < n$ , and thus, by Lemma 2 (for  $\mathbf{K} = R_{\mathbf{C}}(\rho, n) \mathbf{B}^d$ ), we obtain

**Corollary 1** Let  $d \geq 2, n \geq 1, \rho \geq 1$ , and let  $\mathbf{C}$  be an  $\mathbf{o}$ -symmetric convex body in  $\mathbb{E}^d$ . Then

$$R_{\mathbf{C}}(\rho, n)^d \leq \frac{\operatorname{vol}_d(\mathbf{C})n}{\delta_{\text{sep}}(\rho, \mathbf{C})\kappa_d} \leq (R_{\mathbf{C}}(\rho, n) + 2\rho R(\mathbf{C}))^d. \tag{9}$$

**Lemma 3** Let  $n \geq \frac{4^d \delta_{\text{sep}}(\rho, \mathbf{C}) \rho^d R(\mathbf{C})^d}{r(\mathbf{C})^d}$  and  $i = 1, 2, \dots, d - 1$ . Then for  $R = R_{\mathbf{C}}(\rho, n)$ ,

$$\begin{aligned} &M_i((R + \rho R(\mathbf{C})) \mathbf{B}^d) \\ &\leq M_i(\mathbf{B}^d) \left( \frac{\operatorname{vol}_d(\mathbf{C})n}{\delta_{\text{sep}}(\rho, \mathbf{C})\kappa_d} \right)^{\frac{i}{d}} \left( 1 + \frac{2\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}} \right)^i. \end{aligned}$$

*Proof* Set  $t = R + 2\rho R(\mathbf{C})$ . Then the first inequality in (9) yields that

$$R + \rho R(\mathbf{C}) \leq \frac{t - \rho R(\mathbf{C})}{t - 2\rho R(\mathbf{C})} \left( \frac{\operatorname{vol}_d(\mathbf{C})n}{\delta_{\text{sep}}(\rho, \mathbf{C})\kappa_d} \right)^{\frac{1}{d}}.$$

Thus, by the second inequality in (9) and the condition that  $n \geq \frac{4^d \delta_{\text{sep}}(\rho, \mathbf{C}) \rho^d R(\mathbf{C})^d}{r(\mathbf{C})^d} \geq \frac{4^d \delta_{\text{sep}}(\rho, \mathbf{C}) \rho^d R(\mathbf{C})^d \kappa_d}{\operatorname{vol}_d(\mathbf{C})}$ , we obtain that

$$\begin{aligned} \frac{t - \rho R(\mathbf{C})}{t - 2\rho R(\mathbf{C})} &= 1 + \left( \frac{t}{\rho R(\mathbf{C})} - 2 \right)^{-1} \leq 1 + \frac{2\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C}) \kappa_d^{\frac{1}{d}}}{\operatorname{vol}_d(\mathbf{C})^{\frac{1}{d}}} \cdot \frac{1}{n^{\frac{1}{d}}} \\ &\leq 1 + \frac{2\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}. \end{aligned}$$

□

### 3 Proof of Theorem 1

In the proof that follows we are going to use the following special case of the Alexandrov–Fenchel inequality [13]: if  $\mathbf{K}$  is a convex body in  $\mathbb{E}^d$  satisfying  $M_i(\mathbf{K}) \leq M_i(r\mathbf{B}^d)$  for given  $1 \leq i < d$  and  $r > 0$ , then

$$M_j(\mathbf{K}) \leq M_j(r\mathbf{B}^d) \tag{10}$$

holds for all  $j$  with  $i < j \leq d$ . In particular, this statement for  $j = d$  can be restated as follows: if  $\mathbf{K}$  is a convex body in  $\mathbb{E}^d$  satisfying  $M_d(\mathbf{K}) = M_d(r\mathbf{B}^d)$  for given  $d \geq 2$  and  $r > 0$ , then  $M_i(\mathbf{K}) \geq M_i(r\mathbf{B}^d)$  holds for all  $i$  with  $1 \leq i < d$ .

Let  $d \geq 2$ ,  $1 \leq i \leq d - 1$ ,  $\rho \geq 1$ , and let  $\mathbf{Q}$  be the convex hull of the  $\rho$ -separable packing of  $n$  translates of the  $\mathbf{o}$ -symmetric convex body  $\mathbf{C}$  in  $\mathbb{E}^d$  such that  $M_i(\mathbf{Q})$  is minimal and

$$n \geq \frac{4^d d^{4d}}{\delta_{\text{sep}}(\rho, \mathbf{C})^{d-1}} \cdot \left( \rho \frac{R(\mathbf{C})}{r(\mathbf{C})} \right)^d. \tag{11}$$

By the minimality of  $M_i(\mathbf{Q})$  we have that

$$M_i(\mathbf{Q}) \leq M_i(R\mathbf{B}^d + \mathbf{C}) \leq M_i((R + \rho R(\mathbf{C}))\mathbf{B}^d) \tag{12}$$

with  $R = R_{\mathbf{C}}(\rho, n)$ . Note that (12) and Lemma 3 imply that

$$M_i(\mathbf{Q}) \leq \left( 1 + \frac{2\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}} \right)^i M_i(\mathbf{B}^d) \left( \frac{\text{vol}_d(\mathbf{C})}{\delta_{\text{sep}}(\rho, \mathbf{C})\kappa_d} \right)^{\frac{i}{d}} \cdot n^{\frac{i}{d}}.$$

We examine the function  $x \mapsto (1 + x)^i$ , where, by (11), we have  $x \leq x_0 = \frac{1}{2d^4}$ . The convexity of this function implies that  $(1 + x)^i \leq 1 + i(1 + x_0)^{i-1}x$ . Thus, from the inequality  $\left(1 + \frac{1}{2d^4}\right)^{d-1} \leq \frac{33}{32} < 1.05$ , where  $d \geq 2$ , the upper bound for  $M_i(\mathbf{Q})$  in Theorem 1 follows.

On the other hand, in order to prove the lower bound for  $M_i(\mathbf{Q})$  in Theorem 1, we start with the observation that (10) [based on (12)], (11), and Lemma 3 yield that

$$\begin{aligned} S(\mathbf{Q}) &\leq S((R + \rho R(\mathbf{C}))\mathbf{B}^d) \\ &\leq d\kappa_d \left( \frac{n \text{vol}_d(\mathbf{C})}{\delta_{\text{sep}}(\rho, \mathbf{C})\kappa_d} \right)^{\frac{d-1}{d}} \left( 1 + \frac{2\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}} \right)^{d-1}. \end{aligned} \tag{13}$$

Thus, (13) together with the inequalities  $S(\mathbf{Q})r(\mathbf{Q}) \geq \text{vol}_d(\mathbf{Q})$  (cf. [11]) and  $\text{vol}_d(\mathbf{Q}) \geq n \text{vol}_d(\mathbf{C})$  yield

$$r(\mathbf{Q}) \geq \left( 1 + \frac{2\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}} \right)^{-(d-1)} \frac{\text{vol}_d(\mathbf{C})^{\frac{1}{d}} \delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{d-1}{d}}}{d\kappa_d^{\frac{1}{d}}} \cdot n^{\frac{1}{d}}. \tag{14}$$



Applying the assumption (11) and  $\text{vol}_d(\mathbf{C}) \geq \kappa_d r(\mathbf{C})^d$  to (14), we get that

$$r(\mathbf{Q}) \geq \left(1 + \frac{1}{2d^4}\right)^{-(d-1)} \frac{\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})}{d} n^{\frac{1}{d}} \geq \frac{4d^3}{\left(1 + \frac{1}{2d^4}\right)^{d-1}} R(\mathbf{C}) \geq 31R(\mathbf{C}). \tag{15}$$

Let  $\mathbf{P}$  denote the convex hull of the centers of the translates of  $\mathbf{C}$  in  $\mathbf{Q}$ . Then, (15) implies

$$r(\mathbf{P}) \geq r(\mathbf{Q}) - R(\mathbf{C}) \geq \frac{30}{31}r(\mathbf{Q}) \geq \frac{8\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})}{9d} \cdot n^{\frac{1}{d}}. \tag{16}$$

Hence, by (16) and Lemma 2,

$$\text{vol}_d(\mathbf{Q}) \geq \text{vol}_d(\mathbf{P}) \geq \left(1 + \frac{9d\rho R(\mathbf{C})}{4\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{-d} \cdot \frac{n \text{vol}_d(\mathbf{C})}{\delta_{\text{sep}}(\rho, \mathbf{C})}, \tag{17}$$

which implies in a straightforward way that

$$\text{vol}_d(\mathbf{Q}) \geq \left(1 + \frac{9d\rho R(\mathbf{C})}{4\delta_{\text{sep}}(\rho, \mathbf{C})r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{-d} \cdot \frac{n \text{vol}_d(\mathbf{C})}{\delta_{\text{sep}}(\rho, \mathbf{C})}. \tag{18}$$

Note that (10) (see the restated version for  $j = d$ ) implies that  $M_i(\mathbf{Q}) \geq \left(\frac{\text{vol}_d(\mathbf{Q})}{\kappa_d}\right)^{\frac{i}{d}} \kappa_i$ . Then, replacing  $\text{vol}_d(\mathbf{Q})$  by the right-hand side of (18), and using the convexity of the function  $x \mapsto (1+x)^{-i}$  for  $x > -1$  yields the lower bound for  $M_i(\mathbf{Q})$  in Theorem 1.

Finally, we prove the statement about the spherical shape of  $\mathbf{Q}$ , that is, the inequality (1). As in [8], let

$$\theta(d) = \frac{1}{2^{\frac{d+3}{2}} \sqrt{2\pi} \sqrt{d}(d-1)(d+3)} \min \left\{ \frac{3}{\pi^2 d(d+2)2^d}, \frac{16}{(d\pi)^{\frac{d-1}{4}}} \right\}.$$

Using the inequality  $\frac{\kappa_{d-1}}{\kappa_d} \geq \sqrt{\frac{d}{2\pi}}$  (cf. [1]) and (6) of [10], we obtain

$$\left(\frac{S(\mathbf{Q})}{S(\mathbf{B}^d)}\right)^d \left(\frac{\text{vol}_d(\mathbf{B}^d)}{\text{vol}_d(\mathbf{Q})}\right)^{d-1} - 1 \geq \theta(d) \cdot \left(1 - \frac{r(\mathbf{Q})}{R(\mathbf{Q})}\right)^{\frac{d+3}{2}}$$

(see also (5) of [8]). Substituting (13) and (17) into this inequality, we obtain

$$\begin{aligned} & \left(1 + \frac{2\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{d(d-1)} \left(1 + \frac{9d\rho R(\mathbf{C})}{4\delta_{\text{sep}}(\rho, \mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{d(d-1)} \\ & \geq \left(\frac{S(\mathbf{Q})}{S(\mathbf{B}^d)}\right)^d \left(\frac{\text{vol}_d(\mathbf{B}^d)}{\text{vol}_d(\mathbf{Q})}\right)^{d-1}. \end{aligned}$$

By the assumptions  $d \geq 2$  and (11), it follows that

$$4d^2(d-1) \frac{\rho R(\mathbf{C})}{\delta_{\text{sep}}(\rho, \mathbf{C})r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}} \geq \theta(d) \left(1 - \frac{r(\mathbf{Q})}{R(\mathbf{Q})}\right)^{\frac{d+3}{2}}. \quad (19)$$

Note that by [12],  $\frac{1}{\delta_{\text{sep}}(\rho, \mathbf{C})} \leq \frac{2^{\frac{3d}{2}} \cdot \sqrt{\left(\frac{d(d+1)}{2d}\right)}}{(d+1)^{\frac{d}{2}} \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}+1\right)}$ . This and (19) implies (1), finishing the proof of Theorem 1.

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