ON THE HADWIGER NUMBERS OF CENTRALLY SYMMETRIC STARLIKE DISKS

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Abstract. The Hadwiger number \( H(S) \) of a topological disk \( S \) in \( \mathbb{R}^2 \) is the maximal number of pairwise nonoverlapping translates of \( S \) that touch \( S \). A conjecture of A. Bezdek., K. and W. Kuperberg [2] states that this number is at most eight for any starlike disk. A. Bezdek [1] proved that the Hadwiger number of a starlike disk is at most seventy five. In this note, we prove that the Hadwiger number of any centrally symmetric starlike disk is at most twelve.

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1. INTRODUCTION AND PRELIMINARIES

This paper deals with topological disks in the Euclidean plane \( \mathbb{R}^2 \). We make use of the linear structure of \( \mathbb{R}^2 \), and identify a point with its position vector. We denote the origin by \( o \).

A topological disk, or shortly disk, is a compact subset of \( \mathbb{R}^2 \) with a simple, closed, continuous curve as its boundary. Two disks \( S_1 \) and \( S_2 \) are nonoverlapping, if their interiors are disjoint. If \( S_1 \) and \( S_2 \) are nonoverlapping and \( S_1 \cap S_2 \neq \emptyset \), then \( S_1 \) and \( S_2 \) touch. A disk \( S \) is starlike relative to a point \( p \), if, for every \( q \in S \), \( S \) contains the closed segment with endpoints \( p \) and \( q \). In particular, a convex disk \( C \) is starlike relative to any point \( p \in C \). A disk \( S \) is centrally symmetric, if \( -S \) is a translate of \( S \). If \( -S = S \), then \( S \) is o-symmetric.

The Hadwiger number, or translative kissing number, of a disk \( S \) is the maximal number of pairwise nonoverlapping translates of \( S \) that touch \( S \). The Hadwiger number of \( S \) is denoted by \( H(S) \). It is well known (cf. [8]) that the Hadwiger number of a parallelogram is eight, and the Hadwiger number of any other convex disk is six. In [9], the authors showed that the Hadwiger number of a disk is at least six. Recently, Cheong and Lee [4] constructed, for every \( n > 0 \), a disk with Hadwiger number at least \( n \).

A. Bezdek, K. and W. Kuperberg [2] conjectured that the Hadwiger number of any starlike disk is at most eight (see also Conjecture 6, p. 95 in the book [3] of

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Brass, Moser and Pach). The only result regarding this conjecture is due to A. Bezdek, who proved in [1] that the Hadwiger number of a starlike disk is at most seventy five. Our goal is to prove the following theorem.

**Theorem.** Let $S$ be a centrally symmetric starlike disk. Then the Hadwiger number $H(S)$ of $S$ is at most twelve.

In the proof, Greek letters, small Latin letters and capital Latin letters denote real numbers, points and sets of points, respectively. For $u, v \in \mathbb{R}^2$, the symbol $\text{dist}(u, v)$ denotes the *Euclidean distance* of $u$ and $v$. For simplicity, we introduce a Cartesian coordinate system and, for a point $u \in \mathbb{R}^2$ with $x$-coordinate $\alpha$ and $y$-coordinate $\beta$, we may write $u = (\alpha, \beta)$. The *closed segment* (respectively, *open segment*) with endpoints $u$ and $v$ is denoted by $[u, v]$ (respectively, by $(u, v)$). For a subset $A$ of $\mathbb{R}^2$, $\text{int} A$, $\text{bd} A$, card $A$ and conv $A$ denotes the interior, the boundary, the cardinality and the convex hull of $A$, respectively.

Consider a convex disk $C$ and two points $p, q \in \mathbb{R}^2$. Let $[t, s]$ be a chord of $C$, parallel to $[p, q]$, such that $\text{dist}(s, t) \geq \text{dist}(s', t')$ for any chord $[s', t']$ of $C$ parallel to $[p, q]$. The *C-distance* $\text{dist}_C(p, q)$ of $p$ and $q$ is defined as

$$\text{dist}_C(p, q) = \frac{2 \text{dist}(p, q)}{\text{dist}(s, t)}.$$ 

For the definition of $C$-distance, see also [10]. It is well known that the $C$-distance of $p$ and $q$ is equal to the distance of $p$ and $q$ in the normed plane with unit disk $\frac{1}{2}(C - C)$. The $o$-symmetric convex disk $\frac{1}{2}(C - C)$ is called the *central symmetral* of $C$. We note that $C \subset C'$ yields $\text{dist}_C(p, q) \geq \text{dist}_{C'}(p, q)$ for any $p, q \in \mathbb{R}^2$.

We prove the theorem in Section 2. During the proof we present two remarks, showing that as we broaden our knowledge of $S$, we are able to prove better and better upper bounds on its Hadwiger number.

## 2. Proof of the Theorem

Let $S$ be an $o$-symmetric starlike disk. Let $\mathcal{F} = \{S_i : i = 1, 2, \ldots, n\}$ be a family of translates of $S$ such that $n = H(S)$ and, for $i = 1, 2, \ldots, n$, $S_i = c_i + S$ touches $S$ and does not overlap with any other element of $\mathcal{F}$. Let $K = \text{conv } S$, $X = \{c_i : i = 1, 2, \ldots, n\}$, $C = \text{conv } X$ and $\bar{C} = \text{conv } (X \cup (-X))$. Furthermore, let $R_i$ denote the closed ray $R_i = \{\lambda c_i : \lambda \in \mathbb{R} \text{ and } \lambda \geq 0\}$.

First, we prove a few lemmas.

**Lemma 1.** The disk $S$ is starlike relative to the origin $o$. Furthermore, $o \in \text{int } S$.

**Proof.** Let $S$ be starlike relative to $p \in S$, and assume that $p \neq o$. By symmetry, $S$ is starlike relative to $-p$. Consider a point $q \in S$. Since $S$ is starlike relative to $p$ and $-p$, the segments $[p, q]$ and $[-p, q]$ are contained in $S$. Thus, any segment $[p, r]$, where $r \in [-p, q]$, is contained in $S$. In other words, we have $\text{conv } \{p, -p, q\} \subset S$, which yields that $[o, q] \subset S$. The second assertion follows from the first and the symmetry of $S$. □

**Lemma 2.** If $x + S$ and $y + S$ are nonoverlapping translates of $S$, then we have $\text{dist}_K(x, y) \geq 1$.
Proof. Without loss of generality, we may assume that \( x = o \). Suppose that \( y \in \text{int} \, K \). Note that there are points \( p, q \in S \) such that \( y \in \text{int conv} \{ o, p, q \} \). By the symmetry of \( S \), \([y - p, y]\) and \([y - q, y]\) are contained in \( y + S \). Since \( y \in \text{int conv} \{ o, p, q \} \), the segments \([y - p, y]\) and \([o, q]\) cross, which yields that \( S \) and \( y + S \) overlap; a contradiction. Hence, \( y \notin \text{int} \, K \). Since \( \text{int} \, K \) is the set of points in the plane whose distance from \( o \), in the norm with unit ball \( K \), is less than one, we have \( \text{dist}_K(o, y) \geq 1 \). □

Remark 1. The Hadwiger number \( H(S) \) of \( S \) is at most twenty four.

Proof. Note that, for every value of \( i \), \( K \) and \( c_i + K \) either overlap or touch. Since \( K \) is \( o \)-symmetric, it follows that \( c_i \in 2K \), and \( c_i + \frac{1}{2}K \) is contained in \( \frac{1}{2}K \). By Lemma 2, \( \{ c_i + \frac{1}{2}K : i = 1, 2, \ldots, n \} \cup \{ \frac{1}{2}K \} \) is a family of pairwise nonoverlapping translates of \( \frac{1}{2}K \). Thus, \( n \leq 24 \) follows from an area estimate. □

Lemma 3. If \( j \neq i \), then \( R_i \cap \text{int} \, S_j = \emptyset \). Furthermore, \( R_i \cap S_j \subset (o, c_i) \).

Proof. Since \( S \) and \( S_i \) touch, there is a (possibly degenerate) parallelogram \( P \) such that \( \text{bd} \, P \subset (S \cup S_i) \) and \( [o, c_i] \subset P \) (cf. Figure 1). Note that if \( \text{int} \, (x + S) \) intersects neither \( S \) nor \( S_i \), then \( x \notin P \) and \( \text{int} \, (x + S) \cap (o, c_i) = \emptyset \).

![Figure 1](image_url)

If \( S_j \cap R_i = \emptyset \), we have nothing to prove. Let \( S_j \cap R_i \neq \emptyset \) and consider a point \( c_j + p \in S_j \cap R_i \). Since \( o \in \text{int} \, S \), \( c_j + p \neq o \) and \( c_j + p \neq c_i \). By the previous paragraph, if \( c_j + p \in (o, c_i) \), then \( c_j + p \notin \text{int} \, S_j \). Thus, we are left with the case that \( c_j + p \in R_i \setminus [o, c_i] \). By symmetry, \( c_i - p \in S_i \). Note that \( (c_i, c_i - p) \cap (o, c_j) \neq \emptyset \), which yields that \( \text{int} \, S_i \) intersects \( (o, c_j) \); a contradiction. □

Lemma 4. We have \( o \in \text{int} \, C \), and \( X \subset \text{bd} \, C \).

Proof. Assume that \( o \notin \text{int} \, C \). Note that there is a closed half plane \( H \), containing \( o \) in its boundary, such that \( C \subset H \). Let \( p \) be a boundary point of \( S \) satisfying \( S \subset p + H \). Then, for \( i = 1, 2, \ldots, n \), we have \( S_i \subset p + H \). Observe that, for any value of \( i \), \( 2p + S \) touches \( S \) and does not overlap \( S_i \). Thus, \( S \cup \{ 2p + S \} \) is a family of pairwise nonoverlapping translates of \( S \) in which every element touches \( S \), which contradicts our assumption that \( \text{card} \, S = n = H(S) \).
Assume that \( c_i \notin \text{bd} \, C \) for some \( i \), and note that there are values \( j \) and \( k \) such that \( c_i \in \text{int conv} \{ o, c_j, c_k \} \). Since \( S_j \) and \( S_k \) touch \( S_i \), \( \frac{1}{2}c_j \) and \( \frac{1}{2}c_k \) are contained in \( K \). Observe that at least one of \( d_j = c_i - \frac{1}{2}c_j \) and \( d_k = c_i - \frac{1}{2}c_k \) is in the exterior of the closed, convex angular domain \( D \) bounded by \( R_j \cup R_k \) (cf. Figure 2). Since \( d_j \) and \( d_k \) are points of \( c_i + K \), we obtain \( (c_i + K) \setminus D \neq \emptyset \). On the other hand, Lemma 3 yields that \( S_i \subset D \), hence, \( c_i + K = \text{conv} S_i \subset D \); a contradiction. \( \square \)

\[ \text{Figure 2} \]

**Remark 2.** The Hadwiger number \( H(S) \) of \( S \) is at most sixteen.

**Proof.** Golab [7] proved that the circumference of every centrally symmetric convex disk measured in its norm is at least six and at most eight. Fáry and Makai [6] proved that, in any norm, the circumferences of any convex disk \( C \) and its central symmetral \( \frac{1}{2}(C - C) \) are equal. Thus, the circumference of \( C \) measured in the norm with unit ball \( \frac{1}{2}(C - C) \) is at most eight.

Since \( C \subset 2K \), we have \( \text{dist}_C(p, q) \geq \text{dist}_{2K}(p, q) = \frac{1}{2} \text{dist}_K(p, q) \) for any points \( p, q \in \mathbb{R}^2 \). By Lemma 2, \( \text{dist}_K(c_i, c_j) \geq 1 \) for every \( i \neq j \). Thus, \( X = \{ c_i : i = 1, 2, \ldots, n \} \) is a set of \( n \) points in the boundary of \( C \) at pairwise \( C \)-distances at least \( \frac{1}{2} \). Hence, \( n \leq 16 \). \( \square \)

Now we are ready to prove our theorem. By [5], there is a parallelogram \( P \), circumscribed about \( \tilde{C} \), such that the midpoints of the edges of \( P \) belong to \( \tilde{C} \). Since the Hadwiger number of any affine image of \( S \) is equal to \( H(S) \), we may assume that \( P = \{ (\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq 1 \text{ and } |\beta| \leq 1 \} \). Note that the points \( e_x = (1, 0) \) and \( e_y = (0, 1) \) are in the boundary of \( \tilde{C} \).

First, we show that there are two points \( r_x \) and \( s_x \) in \( S \), with \( x \)-coordinates \( \rho_x \) and \( \sigma_x \), respectively, such that \( e_x \in \text{conv} \{ o, 2r_x, 2s_x \} \) and \( \rho_x + \sigma_x \geq 1 \).

Assume that \( e_x = c_i \) for some value of \( i \). Since \( S \) and \( S_i \) touch, there is a (possibly degenerate) parallelogram \( P_i = \text{conv} \{ o, r_x, s_x, c_i \} \) such that \( c_i = r_x + s_x \), \( \{ o, r_x \cup [o, s_x] \} \subset S \) and \( \{ c_i, r_x \} \cup [c_i, s_x] \} \subset S_i \) (cf. Figure 1). Observe that \( c_i \in \text{conv} \{ o, 2r_x, 2s_x \} \) and \( \rho_x + \sigma_x = 1 \). If \( e_x = -c_i \), we may choose \( r_x \) and \( s_x \) similarly.
Assume that $e_x \in (c_i, c_j)$ for some values of $i$ and $j$. Consider a parallelogram $P_i = \text{conv}\{o, r_i, s_i, c_i\}$ such that $c_i = r_i + s_i$, $([o, r_i] \cup [o, s_i]) \subset S$ and $([c_i, r_i] \cup [c_i, s_i]) \subset S_i$. Let $L$ denote the line with equation $x = \frac{1}{2}$. We may assume that $L$ separates $s_i$ from $o$. We define $r_j$ and $s_j$ similarly. If the $x$-axis separates the points $s_i$ and $s_j$, we may choose $s_i$ and $s_j$ as $r_x$ and $s_x$ If both $s_i$ and $s_j$ are contained in the open half plane, bounded by the $x$-axis and containing $c_i$ or $c_j$, say $c_i$, we may choose $r_j$ and $s_j$ as $r_x$ and $s_x$ (cf. Figure 3). If $e_x$ is in $(-c_i, c_j)$ or $(-c_i, -c_j)$, we may apply a similar argument.

![Figure 3](image)

Analogously, we may choose points $r_y$ and $s_y$ in $S$, with $y$-coordinates $\rho_y$ and $\sigma_y$, respectively, such that $e_y \in \text{conv}\{o, 2r_y, 2s_y\}$ and $\rho_y + \sigma_y \geq 1$. We may assume that $\rho_x \leq \sigma_x$ and that $\rho_y \leq \sigma_y$.

Let $Q_1, Q_2, Q_3$ and $Q_4$ denote the four closed quadrants of the coordinate system in counterclockwise cyclic order. We may assume that $X \cap Q_1 \neq \emptyset$, and that $Q_1$ contains the points with nonnegative $x$- and $y$-coordinates. We relabel the indices of the elements of $X$ in a way that $R_1, R_2, \ldots, R_n$ are in counterclockwise cyclic order, and the angle between $R_1$ and the positive half of the $x$-axis, measured in the counterclockwise direction, is the smallest amongst all rays in \{ $R_i : i = 1, 2, \ldots, n$ \}.

If \text{card}(Q_i \cap X) \leq 3$ for each value of $i$, the assertion holds. Thus, we may assume that, say, $j = \text{card}(Q_1 \cap X) > 3$. By Lemma 3, $[c_i, c_i - s_y]$ does not cross the rays $R_1$ and $R_j$ for $i = 2, 3, \ldots, j - 1$. Thus, the $y$-coordinate of $c_i$ is at least $\sigma_y$ (cf. Figure 4, note that $c_i$ is not contained in the dotted region). Similarly, the $x$-coordinate of $c_i$ is at least $\sigma_x$ for $i = 2, \ldots, j - 1$. Thus, $\sigma_x \leq 1$ and $\sigma_y \leq 1$, which yield that $\rho_x \geq 0$ and $\rho_y \geq 0$. Since $\sigma_x \geq 1 - \rho_x$ and $\sigma_y \geq 1 - \rho_y$, each $c_i$, with $2 \leq i \leq j - 1$, is contained in the rectangle $T = \{(\alpha, \beta) \in \mathbb{R}^2 : 1 - \rho_x \leq \alpha \leq 1$ and $1 - \rho_y \leq \beta \leq 1\}$.

Let $B = \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq \rho_x$ and $|\beta| \leq \rho_y\}$. Note that if $S$ and $p + S$ are nonoverlapping and $u, v \in S$, then the parallelogram $\text{conv}\{o, u, v, u + v\}$ does not contain $p$ in its interior. Thus, applying this observation with $\{u, v\} \subset \{\pm r_x, \pm r_y, \pm s_x, \pm s_y\}$, we obtain that $p \notin \text{int}B$ (cf. Figure 5, the dotted parallelograms show the region “forbidden” for $p$).

Furthermore, if $r_x$ and $s_x$ do not lie on the $x$-axis, and $r_y$ and $s_y$ do not lie on the $y$-axis, then the interiors of these parallelograms cover $B$, apart from some points of $S$, and thus, we have $p \notin B$. If $p$ is on a vertical side of $B$, then $r_y$ or $s_y$ lies on the $y$-axis (cf. Figure 6). Note that if $r_y$ lies on the $y$-axis, then $e_y \in \text{conv}\{o, 2r_y, 2s_y\}$ yields $\rho_y \geq \frac{1}{2}$, or that also $s_y$ lies on the $y$-axis. Thus, it follows in this case that
$\frac{1}{2} e_y \in S$. Similarly, if $p$ is on a horizontal side of $B$, then $\frac{1}{2} e_x \in S$. We use this observation several times in the next three paragraphs.

Note that $T = (1 - \frac{\rho_x}{4}, 1 - \frac{\rho_y}{4}) + \frac{1}{2} B$. Since for any $2 \leq i < k \leq j - 1$, $c_i + \frac{1}{2} B$ and $c_k + \frac{1}{2} B$ do not overlap, it follows that $c_i$ and $c_k$ lie on opposite sides of $T$. By Lemma 4, we immediately obtain that $j \leq 5$.

Assume that $j = 5$. Then, we have $\operatorname{card}(X \cap T) = 3$, which implies that two points of $X \cap T$ are consecutive vertices of $T$. Without loss of generality, we may assume that $c_4 = (1 - \rho_x, 1)$, $c_3 = (1, 1)$ and $c_2 = (\tau, 1 - \rho_y)$ for some $\tau \in [1 - \rho_y, 1]$. Since $c_3 - c_4$ lies on a vertical side of $B$, we obtain that $\frac{1}{2} e_y \in S$. From the position of $c_3 - c_2$, we obtain similarly that $\frac{1}{2} e_y \in S$. Thus, if $c_1$ is not on the $x$-axis or $c_5$ is not on the $y$-axis, then $R_1 \cap \operatorname{int} S_2 \neq \emptyset$ or $R_5 \cap \operatorname{int} S_4 \neq \emptyset$, respectively; a contradiction. Hence, from $\frac{1}{2} e_x, \frac{1}{2} e_y \in S$, it follows that $c_1 = e_x$ and $c_5 = e_y$. By Lemma 4, we have that $c_2 = (1, 1 - \rho_y)$, which yields that, for example, $S_1$ and $S_2$ overlap; a contradiction.

We are left with the case $j = 4$. We may assume that $c_2$ and $c_3$ lie, say, on the vertical sides of $T$. Then we immediately have $\frac{1}{2} e_y \in S$. If $c_4$ is not on the $y$-axis,
then $R_4 \cap \text{int} S_3 \neq \emptyset$, and thus, it follows that $c_4 = e_y$. We show, by contradiction, that $\text{card}((Q_1 \cup Q_2) \cap X) \leq 6$.

Assume that $\text{card}((Q_1 \cup Q_2) \cap X) > 6$. Note that in this case $\text{card}(Q_2 \cap X) = 4$, and both $c_5$ and $c_6$ are either on the horizontal sides, or on the vertical sides of $T' = (-2 + \rho_x, 0) + T$. If they are on the horizontal sides, then $\pm e_x \in S$, $c_5 = (-1, 1)$, $c_7 = -e_x$, and, by Lemma 4, $c_6 = (-1, 1 - \rho_y)$. Thus, $S_6$ overlaps both $S_5$ and $S_7$; a contradiction, and we may assume that $c_5$ and $c_6$ are on the vertical sides of $T'$.

Since the $y$-coordinate of $c_2$ is at least $\frac{1}{2}$, and since $(c_3, c_3 - \frac{1}{2} e_y)$ does not intersect the ray $R_2$, we obtain that the $y$-coordinate of $c_3$ is at least $\frac{3}{4}$. Similarly, the $y$-coordinate of $c_5$ is at least $\frac{3}{4}$. Note that $c_3 - s_x$ and $c_5 + s_x$ are on the positive half of the $y$-axis. Then it follows from Lemma 3 that $c_3 - s_x$ and $c_5 + s_x$ lie on the open segment $(o, c_4)$. If $c_3 - s_x \notin (\frac{1}{2} c_4, c_4)$ or $c_5 + s_x \notin (\frac{1}{2} c_4, c_4)$, then we have $c_5 + s_x \notin (o, c_4)$ or $c_3 - s_x \notin (o, c_4)$, respectively. Thus, both $c_5 + s_x$ and $c_3 - s_x$ belong to $(\frac{1}{2} c_4, c_4)$, and a neighborhood of $\frac{1}{2} c_4$ intersects $S_4$ in a segment, which yields that $S_4$ is not a disk; a contradiction.
Assume that \( \text{card}(Q_4 \cap X) > 3 \). Then \( \text{card}((Q_1 \cup Q_4) \cap X) > 6 \) yields that \( \text{card}((Q_3 \cup Q_4) \cap X) \leq 6 \), and the assertion follows. Thus, we may assume that \( \text{card}(Q_4 \cap X) \leq 3 \).

Finally, assume that \( \text{card}(Q_3 \cap X) > 3 \). Then we have \( \text{card}((Q_3 \cup Q_4) \cap X) \leq 6 \) or \( \text{card}((Q_2 \cup Q_3) \cap X) \leq 6 \). In the first case we clearly have \( \text{card}X \leq 12 \). In the second case, by the argument used for \( Q_1 \cap X \), we obtain that \( -e_x \in X \) and \( \text{card}(Q_2 \cap X) \leq 3 \), from which it follows that \( \text{card}((Q_1 \cup Q_2 \cup Q_3) \cap X) \leq 9 \). Since \( \text{card}(Q_4 \cap X) \leq 3 \), the assertion holds.

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