A PARTIAL PROOF OF THE ERDŐS-SZEKERES CONJECTURE FOR HEXAGONS

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Abstract. Erdős and Szekeres [5] made the conjecture that, for \( n \geq 3 \), any set of \( 2^n - 2 + 1 \) points in the plane, in general position, contains \( n \) points in convex position. A computer-based proof of this conjecture for \( n = 6 \) appeared in [9] of Szekeres and Peters. The aim of this paper is to give a proof of the conjecture for \( n = 6 \), without the use of computers, under the restriction that the convex hull of the point set is a pentagon.

1. Introduction

In the early 1930s Esther Klein asked whether there is an integer \( N \), for every \( n \geq 3 \), such that any planar set of \( N \) points in general position contains \( n \) points in convex position. Paul Erdős and George Szekeres [5] showed the existence of such an integer, and also that there is a solution satisfying \( N \leq \binom{2n-4}{n-2} + 1 \). This problem is well-known as the “happy ending problem”.

The task that arose naturally was to find the smallest value \( g(n) \) of \( |S| \) with the mentioned property for each \( S \). In [5], the authors made the following conjecture.

Conjecture 1 (Erdős-Szekeres Conjecture). Let \( n \geq 3 \). Then the smallest number \( g(n) \) such that every planar set of \( g(n) \) points in general position contains \( n \) points in convex position, is \( 2^n - 2 + 1 \).

In [6], Erdős and Szekeres constructed a planar set of \( 2^n - 2 \) points in general position that does not contain \( n \) points in convex position. Presently, the best known bounds are

\[
2^n - 2 + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1.
\]

The upper bound is due to G. Tóth and Valtr [10].

Another attempt is to verify the Conjecture for small values of \( n \). Note that three points in general position are in convex position. Thus, clearly \( g(3) = 3 \). The value of \( g(4) \) was determined by Esther Klein in the early 1930s.

According to [9], Makai was the first to prove the equality \( g(5) = 9 \) but he has never published his result. The first published proof appeared in 1970 in [7]. In 1974, Bonnice [2] gave a simple and elegant proof of the same result. In [1], Bisztriczky and G. Fejes Tóth also mention an unpublished proof by Böröczky and Stahl.

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The case \( n = 6 \) seems considerably more complicated. Bonnice [2] makes the following comparison. In a set of nine points, we have \( \binom{9}{5} = 126 \) possibilities for five points to be in convex position, whereas in a set of seventeen points, we have \( \binom{17}{6} = 12376 \) possibilities for six points to be in convex position.

For this case, a computer-based proof has been given recently by Peters and Szekeres [9]. Peters and Szekeres state that their program proved the case of convex pentagons in less than one second on a 1.5GHz computer, but to check the case of convex hexagons required approximately 3000 GHz hours. For other results related to Theorem 1 in the next section. We note that, using the same tools, it may be shown that every planar set of twenty five points in general position contains six points in convex position. We observe also that, by Lemma 4, our proof yields that both examples the convex hulls of the points are pentagons. We present the proof of Theorem 1 in the next section. We note that, using the same tools, it may be shown that every planar set of twenty five points in general position contains six points in convex position. We observe also that, by Lemma 4, our proof yields that both examples the convex hulls of the points are pentagons. We present the proof of Theorem 1 in the next section. We note that, using the same tools, it may be shown that every planar set of twenty five points in general position contains six points in convex position. We observe also that, by Lemma 4, our proof yields that both examples the convex hulls of the points are pentagons. We present the proof of Theorem 1 in the next section. We note that, using the same tools, it may be shown that every planar set of twenty five points in general position contains six points in convex position. We observe also that, by Lemma 4, our proof yields that both examples the convex hulls of the points are pentagons. We present the proof of Theorem 1 in the next section. We note that, using the same tools, it may be shown that every planar set of twenty five points in general position contains six points in convex position. We observe also that, by Lemma 4, our proof yields that both examples the convex hulls of the points are pentagons. We present the proof of Theorem 1 in the next section.
assume that the points of $W = \bigcup_{i=0}^{m} V(P_i)$ are in general position. If $q + 2t < k$ then $W$ contains a hexagon.

Proof. Let us denote by $X_i$ the set of points that are beyond exactly the edge $[x_i, x_{i+1}]$ of $P_i$, and observe that every vertex of $P_0$ is contained in $X_i$ for some value of $i$. If $\text{card}(X_i \cap V(P_0)) + \text{card}(V(P_i)) \geq 6$ for some $P_i$ then the assertion follows (cf. Figure 2). Since $\text{card}(X_i \cap V(P_0)) + \text{card}(V(P_i)) \leq 5$ for each $P_i$ yields that $k = \text{card}(V(P_0)) \leq 0 \cdot p + 1 \cdot q + 2 \cdot t$, we are done. □

We use Lemma 2 often during the proof with $k = 5$. For simplicity, in such cases we use the notation $P_1 * P_2 * \ldots * P_m$.

**Lemma 3.** Let $S \subset \mathbb{E}^2$ be a set of eleven points in general position such that $P = [S]$ is a pentagon, $Q = [S \setminus V(P)]$ is a triangle, and $[S \setminus (V(P) \cup \{q\})]$ is a quadrilateral for every $q \in V(Q)$. Then $S$ contains a hexagon.

Proof. Let $Q = [q_1, q_2, q_3]$ and $R = [S \setminus (V(P) \cup V(Q))] = [r_1, r_2, r_3]$. Observe that for any $i \neq j$, the straight line $L(r_i, r_j)$ strictly separates the third vertex of $R$ from a unique vertex of $Q$. We may label our points in a way that $q_1$, $q_2$, and $q_3$ are in counterclockwise cyclic order, and $L(r_i, r_j)$ separates $r_k$ and $q_k$ for any $i \neq j \neq k \neq i$. Let us denote by $Q_k$ the open convex domain bounded by $L^-(q_k, r_i)$ and $L^-(q_k, r_j)$ for every $i \neq j \neq k \neq i$. For every $i \neq j$, let us denote by $Q_{ij}$ the open convex domain that is bounded by the rays $L^-(q_i, r_j), L^-(q_j, r_i)$, and the segment $[q_i, q_j]$ (cf. Figure 3).

Observe that if $Q_{12}$ contains at least two vertices of $P$ then these vertices together with $q_1, q_2, r_1$, and $r_2$ are vertices of a hexagon. Similarly, if $Q_1 \cup Q_{13} \cup Q_3$ contains at least three vertices of $P$, or $Q_2 \cup Q_{23} \cup Q_3$ contains at least three vertices of $P$ then $S$ contains a hexagon. Since $P$ is a pentagon, we may assume that $Q_{12}$ contains one, $Q_1 \cup Q_{13}$ and $Q_2 \cup Q_{23}$ both contain two, and $Q_3$ contains no vertex of $P$. By symmetry, we obtain that $S$ contains a hexagon unless $\text{card}(Q_i \cap V(P)) = 0$ and $\text{card}(Q_{ij} \cap V(P)) = 1$ for every $i \neq j$. Since the latter case contradicts the condition that $P$ is a pentagon, $S$ contains a hexagon. □
Lemma 4. Let \( S \subset \mathbb{E}^2 \) be a set of thirteen points in general position such that \( P = |S| \) is a pentagon and \( Q = |S \setminus V(P)| \) is a triangle. Then \( S \) contains a hexagon.

Proof. Let \( q_1, q_2, \) and \( q_3 \) be the vertices of \( Q \) in counterclockwise cyclic order and let \( R = S \setminus (V(P) \cup V(Q)) \). Observe that \( \text{card } R = 5 \). Using an idea of Klein and Szekeres, we obtain that \( R \) contains an empty quadrilateral. In other words, there is a quadrilateral \( U \) that satisfies \( V(U) \subset R \) and \( U \cap R = V(U) \).

We show that if \( U \) has no sideline that separates \( U \) from an edge of \( Q \) then \( S \) contains a hexagon. Indeed, if every sideline of \( U \) separates \( U \) from exactly one vertex of \( Q \) then, by the pigeon-hole principle, \( Q \) has a vertex, say \( q_3 \), such that at least two sidelines of \( U \) separate \( U \) from it. This yields that there are two sidelines passing through consecutive edges of \( U \) that separate \( U \) from only \( q_3 \). Let these edges be \( [r_{i-1}, r_i] \) and \( [r_i, r_{i+1}] \). Then we have \( [q_1, r_{i+1}, r_i, r_{i-1}, q_2] * [q_2, r_{i-1}, q_3] * [q_3, r_i, q_1] \). Hence, we may assume that \( U \) has a sideline that separates \( U \) from an edge of \( Q \). Without loss of generality, let this sideline pass through the edge \( [r_1, r_2] \) and let it separate \( U \) from \( [q_1, q_2] \).

For every \( 3 \neq i \neq j \neq 3 \), let \( x_i, y_i, \) and \( z_i \) denote the intersection point of the segment \([q_i, q_3]\) with the line \( L(q_i, r_j) \), \( L(q_j, r_i) \), and \( L(r_i, r_2) \), respectively, and let \( w_i \) denote the intersection point of \([r_i, q_3]\) and \( L(q_j, r_j) \) (cf. Figure 4). If some point \( u \in R \) is beyond exactly the edge \([r_1, r_2]\) of \([q_1, q_2, r_2, r_1]\) then we have \([q_1, r_1, u, r_2, q_2, q_3, r_1, q_1] \). If \( u \in R \) is beyond exactly the edge \([q_1, q_3]\) of \([q_1, r_1, q_3]\) then \([q_1, r_1, u, q_3] \). If \( u \in R \) is beyond exactly the edge \([r_1, q_3]\) of \([q_1, r_1, q_3]\) then \([q_1, r_1, u, q_3] \). If \( u \in R \) is beyond exactly the edge \([q_1, q_3]\) of \([q_1, r_1, q_3]\) then \([q_1, r_1, u, q_3] \).

Assume that \( r_3 \in [r_1, w_1, x_1, y_1] \). If \( L^+(r_4, r_3) \cap [q_1, q_3] \neq \emptyset \) then \([q_3, r_3, r_4, r_1, r_2] \). If \( L^-(r_4, r_3) \cap [q_1, q_3] \neq \emptyset \) then \([r_1, r_3, r_4, q_3] \). If \( L(r_4, r_3) \cap [q_1, q_3] = \emptyset \) then \([q_1, r_1, r_2, q_3, q_3, r_4, q_3] \). If \( L(r_4, r_3) \cap [q_1, q_3] = \emptyset \) then \([q_1, r_1, r_2, q_3, q_3, r_4, q_3] \). Thus, we may assume that \( r_3 \in [r_2, z_2, x_2, w_2] \). Since \( r_3 \in [r_2, y_2, z_2] \) yields \([q_3, r_4, r_1, r_2, r_3] + [r_3, r_2, q_1] \) \([q_1, r_4, q_3] \), we may assume that \( r_3 \in [r_2, w_2, x_2, y_2] \), and (by symmetry) that \( r_4 \in [r_1, w_1, x_1, y_1] \).

Assume that \( r \in [r_1, w_1, x_1, y_1] \). If \([r_1, r_2, r_4, r] \) is a quadrilaterals then we may apply an argument similar to that in the previous paragraph. Thus, we may assume that \( r_4 \in [r_1, r_2, r] \). This yields \([r, r_4, r_1, q_1] * [q_1, r_1, r_2, q_3] * [q_3, r_3, r_4, r] \). Hence, \( r \in [q_1, q_2, z_2, z_1] \).

If \( r \in [q_1, r_1, z_1] \) then \([q_1, r, r_1, r_2, q_2] * [q_2, r_2, q_3] * [q_3, r_1, q_1] \). Let \( r \in [q_1, q_2, r_2, r_1] \). If \( L(q_3, r_4) \) does not separate \( q_1 \) and \( r \) then \([q_1, r, q_2] * [q_2, r_1, r_4, q_3] * [q_3, r_4, r, q_1] \). Otherwise, we may suppose that \( L^+(r, r_4) \cap [q_1, q_3] \neq \emptyset \). By symmetry, we obtain also that \( L^+(r, r_3) \cap [q_2, q_3] \neq \emptyset \).

Assume that \( r \in [q_1, r_1, r_2] \). Then we observe that \( U' = [r, r_2, r_3, r_1] \) is an empty quadrilateral, and \( L(r, r_2) \) separates \( U' \) from \([q_1, q_2] \). Since \( R \cap [q_1, r, r_2, q_2] = \emptyset \), an argument applied for \( U' \), similar to that applied for \( U \), yields a hexagon. Hence,
r ∈ [q_1, r_1, q_2] ∩ [q_1, r_2, q_2]. Then \( L^+(r_3, r) \cap [q_1, q_2] \neq \emptyset \neq L^+(r_4, r) \cap [q_1, q_2] \). Now, we apply Lemma 3 with \( V(P) \cup V(Q) \cup \{r_3, r_4\} \) as \( S \).

**Definition 1.** Let \( A, B \subset \mathbb{E}^2 \) be sets of points in general position. Suppose that there is a bijective function \( f : A \rightarrow B \) such that, for any \( a_1, a_2, a_3 \in A \), the ordered triples \((a_1, a_2, a_3)\) and \((f(a_1), f(a_2), f(a_3))\) have the same orientation. Then we say that \( A \) and \( B \) are identical.

We note that if \( A \) and \( B \) are identical then \( A' \subset A \) is a \( k \)-gon if, and only if, \( f(A') \) is a \( k \)-gon.

Let \( \tilde{S} \) be a set of less than thirteen points such that \([\tilde{S}]\) is a pentagon, \([\tilde{S} \setminus V(\tilde{S})]\) is a triangle, and \( \tilde{S} \) does not contain a hexagon. Using Lemma 4 and a little more effort, we may characterize the possible configurations for \( \tilde{S} \setminus V(\tilde{S}) \). Lemma 5 summarizes our work. We omit its straightforward proof.

**Lemma 5.** Let \( \tilde{S} \subset \mathbb{E}^2 \) be a set of less than thirteen points in general position such that \([\tilde{S}]\) is a pentagon, \([\tilde{S} \setminus V(\tilde{S})]\) is a triangle, and \( \tilde{S} \) does not contain a hexagon. Then \( Q \) is identical to one of the sets in Figure 5.

This list helps us to exclude some other cases from our investigation. If a set is identical to one of the sets in Figure 5, we say that its type is the type of the corresponding set in the figure.

**Lemma 6.** Let \( S \subset \mathbb{E}^2 \) be a set of seventeen points in general position such that \( P = [S] \) is a pentagon and \( Q = [S \setminus V(P)] \) is a quadrilateral. Then \( S \) contains a hexagon.
Proof. By Lemma 4 we may assume that any diagonal of \( Q \) divides \( Q \) into two triangles that contain exactly four points of \( S \) in their interiors. Furthermore, both these triangles have to be either type 4a, 4b, or 4c. Let us observe that if both triangles contain a pair of points such that the line passing through them does not intersect the diagonal then these two pairs of points and the two endpoints of the diagonal are in convex position. Hence, we may assume that, in at least one of the triangles, each line passing through two points intersects the diagonal.

Since there is, in a type 4c set, no edge of the convex hull that meets all the lines that pass through two of its points, we may assume that the set of the points in one of the triangles is type 4a or 4b, and that the diagonal is the left edge of one of the triangles in Figure 5. We observe also that configurations of type 4a or 4b are almost identical, the only difference is that the line passing through the two points closest to the left edge of the triangle intersects the bottom or the right edge of the triangle. Thus, we may handle these two cases together if we leave it open which edge this line intersects.

We denote our points as in Figure 6, and let \( L = L(r_1, r_2) \). Observe that \( L \) divides the set of points, beyond exactly the edge \( [q_1, q_2] \) of \( [q_1, q_2, q_3] \), into two connected components. If a point \( p \) is in the component that contains \( q_1 \), respectively \( q_2 \), in its boundary then we say that \( p \) is on the left-hand side, respectively right-hand side, of \( L \). Let \( B = (Q \setminus [q_1, q_2, q_3]) \cap S \). Observe that \( \text{card}\, B = 5 \) and that every point of \( B \) is either on the left-hand side or on the right-hand side of \( L \). By the pigeon-hole principle, there are three points of \( B \) that are on the same side of \( L \). Let us denote these points by \( s_1, s_2, \) and \( s_3 \).

Assume that \( s_1, s_2, \) and \( s_3 \) are on the left-hand side of \( L \). Observe that if \( L(s_i, s_j) \) and \( [q_1, r_1] \) are disjoint for some \( i \neq j \) then \( [q_1, s_i, s_j, r_1, r_2, r_3] \) is a hexagon. Thus, we may relabel \( s_1, s_2, \) and \( s_3 \) such that \( s_3 \in [q_1, r_1, s_2] \subset [q_1, r_1, s_1] \). This yields that either \([s_1, s_2, s_3, q_1]\) or \([s_1, s_2, s_3, r_1]\) is a quadrilateral. If \([s_1, s_2, s_3, q_1]\) is a quadrilateral then \([s_1, s_2, s_3, q_1] \ast [q_1, s_3, r_1, r_2, r_3] \ast [r_3, r_4, q_2] \ast [q_2, r_1, s_2, s_1]\). If \([s_1, s_2, s_3, r_1]\) is a quadrilateral then \([s_1, s_2, s_3, r_1, q_2] \ast [q_2, r_4, r_3] \ast [r_3, r_2, r_1, s_3, q_1] \ast [q_1, s_2, s_1]\).

Let \( s_1, s_2, \) and \( s_3 \) be on the right-hand side of \( L \). Observe that if \( L(s_i, s_j) \) and \([q_2, r_1]\) are disjoint for some \( i \neq j \) then \([q_2, s_i, s_j, r_1, r_2, r_3]\) is a hexagon. Hence, we may assume that \( s_3 \in [q_2, r_1, s_2] \subset [q_2, r_1, s_1] \). Then \([s_1, s_2, s_3, q_2]\) or \([s_1, s_2, s_3, r_1]\) is a quadrangle. If \([s_1, s_2, s_3, q_2]\) is a quadrangle then \([s_1, s_2, s_3, q_2] \ast [q_2, s_3, r_1, r_2, r_4] \ast [r_4, r_2, q_1] \ast [q_1, r_1, s_2, s_1]\). If \([s_1, s_2, s_3, r_1]\) is a quadrangle then \([s_1, s_2, s_3, r_1, q_1] \ast [q_1, r_2, r_4] \ast [r_4, r_2, r_1, s_3, q_2] \ast [q_2, s_2, s_1]\). □

Lemma 7. Let \( S \subset \mathbb{E}^2 \) be a set of points in general position such that \( P = [S] \) and \( Q = [S \setminus V(P)] \) are pentagons, and \( S \setminus (V(P) \cup V(Q)) \) has a subset of type 3a, or a subset identical to the point set in Figure 8, 9, or 10. Then \( S \) contains a hexagon.
Proof. Let $R$ denote the subset of $S \setminus (V(P) \cup V(Q))$ that is either of type 3a, or is identical to the point set in Figure 8, 9 or 10. Let $q_1, q_2, q_3, q_4,$ and $q_5$ denote the vertices of $Q$ in counterclockwise cyclic order.

Assume that $R$ is of type 3a. Let us denote the points of $R$ as in Figure 7. Let $R_{12}, R_{23},$ and $R_{13}$ denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_2, t_2, t_3, r_1]$, the edge $[r_2, r_3]$ of $[r_2, t_1, t_3, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_3, r_3]$. If $\text{card}(R_{12} \cap V(Q)) \geq 2$, $\text{card}(R_{23} \cap V(Q)) \geq 2$, or $\text{card}(R_{13} \cap V(Q)) \geq 3$ then $S$ contains a convex hexagon. Otherwise, there is a vertex $q_i$ of $Q$ in the convex domain bounded by the half-lines $L^-((r_2, t_1))$ and $L^-((r_2, t_2))$, from which we obtain $[r_1, t_1, r_2, q_i] \ast [q_i, r_2, t_2, r_3] \ast [r_3, t_3, r_1]$.

Let us assume that $R$ is the set in Figure 8 and denote the points of $R$ as indicated. Let $R_{12}, R_{23},$ and $R_{13}$ denote, respectively, the set of points that are beyond exactly the edge $[r_1, r_2]$ of $[r_1, t_1, t_2, r_2]$, the edge $[r_2, r_3]$ of $[r_2, t_2, t_3, r_3]$, and the edge $[r_1, r_3]$ of $[r_1, t_1, t_3, r_3]$. If $\text{card}(R_{12} \cap V(Q)) \geq 2$, $\text{card}(R_{23} \cap V(Q)) \geq 3$, or $\text{card}(R_{13} \cap V(Q)) \geq 3$ then $S$ contains a hexagon. Hence, we may assume that $q_1 \in R_{12}$, $\{q_2, q_3\} \subset R_{23}$, $\{q_4, q_5\} \subset R_{13}$, and there is no vertex of $Q$ in $R_{23} \cap R_{13}$. If $L(q_1, q_4)$ does not intersect the interior of $[R]$ then the convex hull of $[r_1, t_2, r_2, r_1]$ and $[r_4, q_1]$ is a hexagon.

Let $r_4 \in [q_1, r_1, r_2]$. If $L(r_4, r_1)$ does not separate $q_5$ and $q_1$, and $L(r_4, r_2)$ does not separate $q_2$ and $q_1$ then $[q_1, r_4, r_2, q_2] \ast [q_2, r_2, t_2, r_3] \ast [r_3, t_1, r_1, q_5] \ast [q_5, r_1, r_4, q_1]$. Thus, we may assume that, say, $L(r_4, r_1)$ separates $q_5$ and $q_1$. If $L(r_4, r_3)$ separates $q_4$ and $R$ then $[q_4, r_3, r_2] \ast [r_2, t_2, t_1, r_1] \ast [r_1, t_1, r_3, q_4]$. If $L(r_4, r_3)$ does not separate $q_4$ and $R$ then $[r_4, r_2, r_3, q_4, q_5, r_1]$ is a hexagon.

Assume that $R$ is the set in Figure 9 and denote the points of $R$ as indicated. We may clearly assume that there is no vertex of $Q$ beyond exactly the edge $[r_1, r_2]$ of $[r_1, r_2, t_3, t_1]$. Hence, there is an edge, say $[q_1, q_2]$, that intersects both rays $L^-(r_1, t_1)$ and $L^-(r_2, t_2)$. If $L(r_1, r_2)$ separates $R$ from both $q_1$ and $q_2$ then $[q_1, r_1, r_2, q_2] \ast [q_2, r_2, t_2, r_3] \ast [r_3, t_2, t_1, r_1] \ast [r_4, t_1, r_1, q_1]$. Hence, we may assume that $L(r_1, r_2)$ does not separate $R$, say, from $q_2$. If $L(t_2, t_3)$ does not separate $r_2$ and $q_2$ then $[r_1, r_2, q_2, t_2, t_3, t_1]$ is a hexagon. If $L(t_2, t_3)$ separates $r_2$ and $q_2$ then $[q_1, r_2, q_2] \ast [q_2, t_2, t_3, r_4] \ast [r_4, t_1, r_1, q_1]$.

We are left with the case when $R$ is the set in Figure 10 with points as indicated. Let $R_{12}, R_{23}, R_{34},$ and $R_{14}$ denote, respectively, the set of points that are beyond...
exactly the edge \([r_1, r_2]\) of \([r_1, r_2, t_2, t_1]\), the edge \([r_2, r_3]\) of \([r_2, t_2, t_3, r_3]\), the edge \([r_3, r_4]\) of \([r_3, t_3, t_1, r_4]\), and the edge \([r_4, t_4]\) of \([r_4, t_1, r_4]\). If \(\text{card}(R_{i+1}) \cap V(Q)) \geq 2\) for some \(i \in \{1, 2, 3\}\) then \(S\) contains a hexagon. Otherwise, \(R_{14}\) contains at least two vertices of \(Q\), which we denote by \(q_1\) and \(q_2\). If both \(q_1\) and \(q_2\) are beyond exactly the edge \([r_1, r_4]\) of \([r_1, t_2, t_3, r_4]\) then \([t_2, t_3, r_4, q_1, q_2, r_1]\) is a hexagon. Thus, we may assume that, say, \(q_1\) is beyond exactly the edge \([r_3, r_4]\) of \([r_3, t_3, r_4]\). From this, it follows that \([r_3, t_3, r_4, q_1] \ast [q_1, r_4, t_1, r_1] \ast [r_1, t_1, t_2, r_2] \ast [r_2, t_2, t_3, r_3]\).

\[\square\]

![Figure 9](image1.png)

![Figure 10](image2.png)

Now we are ready to prove Theorem 1.

Let \(Q = [S \setminus V(P)], R = [S \setminus (V(P) \cup V(Q))]\), and \(T = S \setminus (V(P) \cup V(Q) \cup V(R))\). If \(Q\) is a triangle then we apply Lemma 4. If \(Q\) is a quadrilateral, we apply Lemma 6. Let \(Q\) be a pentagon. If \(R\) is a triangle or a quadrilateral then it contains a subset identical to \(S \setminus (V(P) \cup V(Q))\) in Lemma 7. Let \(R\) be a pentagon. We note that \(T\) contains two points, say, \(t_1\) and \(t_2\).

Let \(q_1, q_2, q_3, q_4, q_5\) and \(r_1, r_2, r_3, r_4, r_5\) denote, respectively, the vertices of \(Q\) and \(R\) in counterclockwise cyclic order. If some \(q_i\) is beyond exactly one edge of \(R\) then \([R, q_i]\) is a hexagon. Thus, we may assume that every vertex of \(Q\) is beyond at least two edges of \(R\). Observe that there is no point on the plane that is beyond all five edges of \(R\). If some \(q_i\) is beyond all edges of \(R\) but one, say \([r_1, r_5]\), then we obtain \([r_1, r_2, r_3, r_4, r_5] \ast [r_5, q_4, r_4] \ast [q_4, r_2, r_1]\). Hence, we may assume that every vertex of \(Q\) is beneath at least two edges of \(R\).

For \(1 \leq i \leq 5\), let \(R_i\) denote the set of points that are beyond the two edges of \(R\) that contain \(r_i\) and beneath the other three edges of \(R\), and let \(R_{i(i+1)}\) denote the set of points that are beyond the edges of \(R\) that contain \(r_i\) or \(r_{i+1}\), and beneath the other two edges of \(R\) (cf. Figure 11). We call \(R_{i(i-1)}\) and \(R_{i(i+1)}\) consecutive regions.
Assume that two distinct and nonconsecutive regions contain vertices of $Q$, say, $q_k \in R_{31}$ and $q_l \in R_{23}$. Since every vertex of $Q$ is beneath at least two edges of $R$, $q_k$ and $q_l$ are distinct points. If there is a vertex $q_h$ of $Q$ in $R_{34} \cup R_4 \cup R_{45}$ then \([q_h, r_4, r_5, q_k] * [q_h, r_1, r_2, q_l]\). Let $V(Q) \cap (R_{34} \cup R_4 \cup R_{45}) = \emptyset$. Then exactly one edge of $Q$ intersects $R_{34} \cup R_4 \cup R_{45}$. Let us denote this edge by \([y_m, q_{m+1}]\). If $q_m \in R_{23}$ then \([q_{m+1}, r_4, q_m] * [q_m, r_2, r_1] * [r_1, r_2, r_3, r_4, q_{m+1}]\). Let $q_m \in R_3$ and, by symmetry, \(q_{m+1} \in R_3\). If there are at least three vertices of $Q$ in $R_2 \cup R_{23} \cup R_3$ or in $R_1 \cup R_{15} \cup R_5$ then $V(Q) \cup V(R)$ contains a hexagon.

Hence, we may assume that a vertex $q_g$ of $Q$ is in $R_{12}$. Since every vertex of $Q$ is beneath at least two edges of $Q$, the sum of the angles of $R$ at $r_1$ and $r_2$ is greater than $\pi$, which implies that $L(r_1, r_2)$ separates $R$ and $q_g$. Thus, we have \([q_g, r_2, r_3, q_m] * [q_m, r_4, q_{m+1}] * [q_{m+1}, r_5, r_1, q_3]\).

Assume that two consecutive regions contain vertices of $Q$, say $q_k \in R_{31}$ and $q_l \in R_{12}$. If $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) = \emptyset$ then we may apply the argument in the previous paragraph. Let $V(Q) \cap (R_{23} \cup R_{34} \cup R_{45}) = \emptyset$. If at least four vertices of $Q$ are beneath the edge $[r_3, r_4]$ of $R$ then these vertices, together with $r_3$ and $r_4$, are six points in convex position. Hence, we may assume that $R_3 \cup R_4$ contains at least two vertices of $Q$. Let us denote these vertices by $q_e$ and $q_f$. If $q_e, q_f \in R_4$ then \([r_1, r_2, q_e, q_f, r_4, r_5]\) is a hexagon. Thus, we may clearly assume that, say, $q_e \in R_3$ and $q_f \in R_4$. Then we have \([q_1, r_2, r_3, q_e] * [q_e, r_3, r_4, q_f] * [q_f, r_4, r_5, q_k] * [q_k, r_5, r_1, q_1]\).

Assume that $R_{i(i+1)}$ contains a vertex of $Q$ for some $i$, say $q_1 \in R_{51}$. By the preceding, no vertex of $Q$ is in $R_{12} \cup R_{23} \cup R_{34} \cup R_{45}$. An argument similar to that used in the previous paragraph yields the existence of a hexagon if $R_2, R_3$, or $R_4$ contains no vertex of $Q$. Let $q_k \in R_2, q_l \in R_3$, and $q_m \in R_4$. Then \([q_1, r_1, r_2, q_k] * [q_k, r_2, r_3, q_l] * [q_l, r_3, r_4, q_m] * [q_m, r_4, r_5, q_1]\).

We have now arrived at the case that each vertex of $Q$ is beyond exactly two edges of $R$. Clearly, we may assume that $q_e \in R_i$ for each $i$.

If $L(t_1, t_2)$ intersects two consecutive edges of $R$ then $S$ contains a hexagon. Hence, we may assume that, say, $L^+(t_1, t_2) \cap [r_2, r_3] \neq \emptyset$ and $L^-(t_1, t_2) \cap [r_5, r_1] \neq \emptyset$ (cf. Figure 12). If both $q_1$ and $q_2$ are beyond exactly the edge $[r_1, r_2]$ of $[r_1, t_1, t_2, r_2]$ then we have a hexagon. If neither point is beyond exactly that edge then \([q_1, r_1, r_2, q_2] * [q_2, r_2, t_2, r_3] * [r_3, t_2, t_1, r_5] * [r_5, t_1, r_1, q_1]\). Thus, we may assume that $q_1$ is beyond exactly the edge $[r_1, r_2]$ and $q_2$ is not. If $q_5$ is beyond exactly the edge $[r_4, r_5]$ of $[r_4, r_5, t_1, t_2, r_3]$ then \([q_5, r_5, t_1, t_2, r_3, r_4]\) is a hexagon. Hence, we may assume that $q_5$ is beyond exactly the edge $[r_1, r_5]$ of $[r_1, t_2, r_3]$ and, similarly, that $q_1$ is beyond exactly the edge $[r_2, r_3]$ of $[r_2, t_2, r_3]$. From this, we obtain that $[q_3, r_3, r_4, q_4] * [q_4, r_4, r_5, q_5] * [q_5, r_5, r_1, q_1] * [q_1, r_1, r_2, q_2] * [q_2, r_2, t_2, r_3, q_3].$
References


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