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# Elliptical Orbit $\Rightarrow$ $1/r^2$ Force

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**N**ewton's proof of the connection between elliptical orbits and inverse-square forces ranks among the "top ten" calculations in the history of science. This time-honored calculation is a highlight in an upper-level mechanics course. It would be worthwhile if students in introductory physics could prove the relation *elliptical orbit*  $\Rightarrow$   $1/r^2$  *force* without having to rely on upper-level mathematics. We introduce a simple procedure—Newton's Recipe—that allows students to readily and accurately deduce the algebraic form of force laws from a geometric analysis of orbit shapes.

Newton's Recipe is based on a hidden gem in Newton's *Principia*—the "PQRST Formula," which is a simple geometric version of  $F = ma$ . Given *any* kind of orbital curve (elliptical, spiral, etc.), this formula allows one to deduce the force simply by measuring the lengths of three line segments—the "shape parameters" of the orbit. There are no differential equations or computational programs.

In our "Orb Lab," students solve the celebrated *Kepler Problem*: Given an ellipse, find the force. Students draw a large ellipse, cut it into small pieces (parabolic arcs), measure the force at a point on each arc, and see how the force varies along the orbit. In essence, the class discovers one of the most fundamental laws of nature—the law of gravity—using string, tacks, and a ruler, along with a little help from Galileo, Newton, and Kepler.

## Force and Geometry

There exists a deep connection between force and

geometry. A constant force causes a body to move in a parabolic path. A constant centripetal force causes a body to move in a circular path. In 1609, Johannes Kepler reported that the planet Mars moves in an elliptical orbit. What kind of force causes a planet to move in an elliptical path? What is the force law—the law that specifies how the force  $F(r)$  depends on the distance  $r$  between the Sun and the planet? This Kepler problem challenged the natural philosophers of the 17th century. In general, there are two kinds of problems in orbital mechanics:

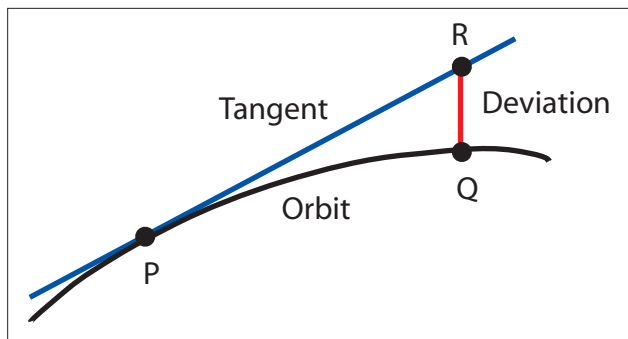
*Direct Problem:* Given the orbit shape, find the force law.

*Inverse Problem:* Given the force law, find the orbit shape.

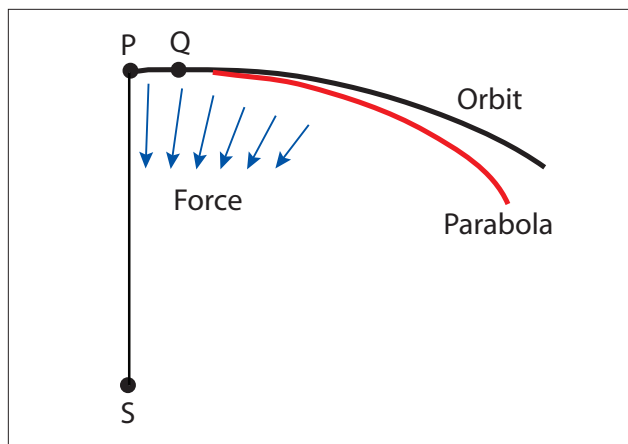
Isaac Newton solved these problems in his *Mathematical Principles of Natural Philosophy*, published in 1687.<sup>1</sup> Table I displays some of Newton's results. In our introductory course, students solve the direct problem.

**Table I. Force-geometry problems solved by Newton.**

Orbit Shape	"Sun" Location	Force Law
Circle	Circumference	$1/r^5$
Spiral	Pole	$1/r^3$
Ellipse	Center	$r$
Ellipse	Focus	$1/r^2$



**Fig. 1.** The force that deflects a planet away from the inertial path PR toward the Sun (below P) is measured by the deviation QR of the actual curved orbit PQ from the virtual straight tangent.



**Fig. 2.** During an infinitesimal time, the orbital arc PQ is approximately parabolic and the variable force is approximately constant. The magnitude of the force depends on the Sun-planet distance  $SP \approx SQ$ . The direction of the force is parallel to SP.

## Geometric Measure of Force

One of the most important formulas in Newton's *Principia* is the formula that relates centripetal force and orbit geometry. This formula is the basis for Newton's orbital mechanics.<sup>2</sup> We present a modern, pedagogical view of Newton's geometric measure of force.

Consider a planet orbiting the Sun. In a certain time interval, the planet moves from point P to point Q along the orbit as shown in Fig. 1. If no force acted on the planet, then the planet would move along the tangent line PR with the constant velocity it had at P. Because of the force, the planet moves along the curved path PQ. The deviation QR of the curved orbit from the straight tangent provides a measure of the force.

What is the mathematical relation between the force and the deviation? The crucial insight of Newton was to realize that for small time intervals, the *variable* force can be treated as a *constant* force in both magnitude and direction. This crucial "parabolic approximation" is illustrated in Fig. 2. In effect, by focusing on tiny parts of the whole orbit, Newton replaced the complex ellipses of Kepler with the simple parabolas of Galileo.

Given the equivalence between projectile motion and short-time orbital motion, we can readily derive the relation between the force  $F$  and the deviation  $d$ . The orbital motion can be viewed as a combination of two independent component motions: inertial motion due to the constant velocity alone (no force) and falling motion due to the constant force alone (no velocity).<sup>3</sup> The continual deviation  $d$  of the orbit from the tangent due to the constant force coincides with the uniformly accelerated falling motion of the planet. During a time  $t$  (infinitesimal), the distance the planet falls is

$$d = \frac{1}{2} at^2. \quad (1)$$

The constant acceleration  $a$  of the planet of mass  $m$  is related to the constant force  $F$  via Newton's law of motion:

$$F = ma. \quad (2)$$

Combining the kinematic relation in Eq. (1) with the dynamic law in Eq. (2) gives

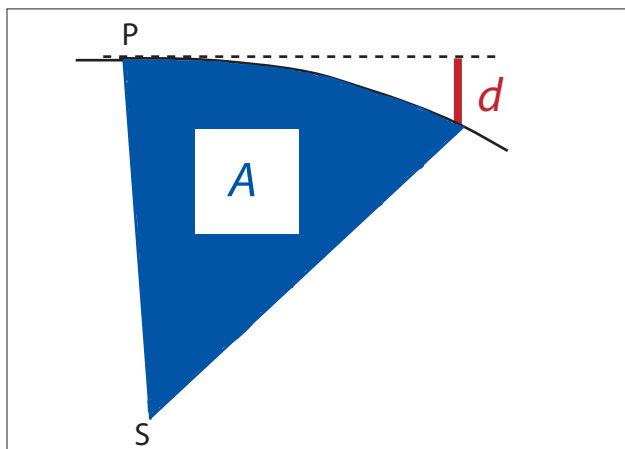
$$F = 2m d/t^2. \quad (3)$$

The important content of the force formula in Eq. (3) is the proportional relation

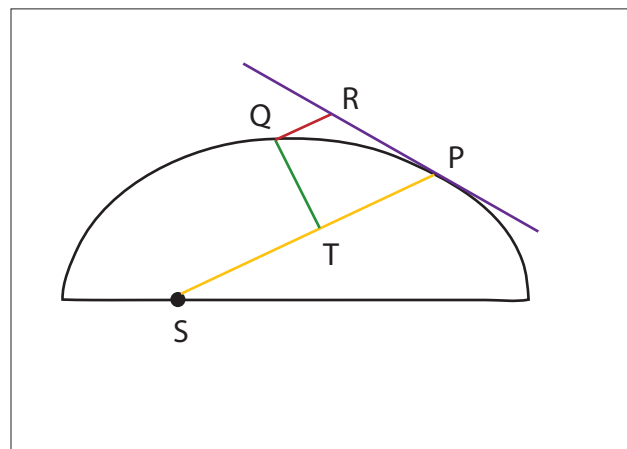
$$F \propto d/t^2. \quad (4)$$

The geometric measure of force, *force*  $\propto$  *deviation of orbit*, is the forerunner of the modern algebraic notion of force, *force*  $\propto$  *change in momentum* (for a given time interval).

Our goal is to find a formula that measures the force on the planet solely from a diagram of the orbit. To convert Eq. (4) into a purely geometric formula,



**Fig. 3.** The geometric measure of force,  $F \propto d/A^2$ , depends on the deviation  $d$  of the orbit from the tangent line and the area  $A$  swept out by the radial line. The limiting value of  $d/A^2$  (for infinitesimal  $d$  and  $A$ ) provides an exact measure of the force at point P due to the Sun S.



**Fig. 4.** Newton's diagram for measuring the force on a planet at point P of the orbit. Point Q is any future point. The radial line SP connects the Sun S to the planet. The deviation line QR separates the orbital arc PQ from the inertial tangent PR and is parallel to SP (the force direction). The time line QT is the height of the "time triangle" SPQ and is perpendicular to SP.

we need to know where to place the temporal parameter  $t$  in the orbital diagram. How does the *shape of the orbit* provide information on the *time of transit*?

The geometric measure of time is provided by Kepler's law of areas. This law says that the line connecting the Sun and the planet sweeps out equal areas in equal times. In other words, the time  $t$  that elapses is proportional to the area  $A$  that is swept out:  $t \propto A$ . Given this time-area relation, we can write Eq. (4) as

$$F \propto d/A^2. \quad (5)$$

Equation (5) is illustrated in Fig. 3.

In summary, the geometric measure of force,  $F \propto d/A^2$ , stems from four elementary principles: (1) parabolic approximation:  $F$  is constant for small  $t$  and  $d$ , (2) kinematic relation:  $d = \frac{1}{2}at^2$  (Galileo), (3) dynamic law:  $F = ma$  (Newton), and (4) area law:  $t \propto A$  (Kepler).

## Newton's Force Formula

Newton's version<sup>4</sup> of the basic force formula,  $F \propto d/A^2$  ( $d \rightarrow 0$ ), is

$$F \propto QR/(SP \times QT)^2 \quad (Q \rightarrow P). \quad (6)$$

The force at point P depends on the length of three line segments (QR, SP, QT) that characterize the shape of the orbit around P. These three *shape parameters* are defined in Fig. 4. As we have seen, the deviation  $d$  of the orbit from the tangent is equal to QR. In the limit  $Q \rightarrow P$ , the area  $A$  of the sector swept out during the motion from P to Q is equal to the area of the triangle SPQ as shown in Fig. 4. The triangle SPQ has base SP, height QT, and area  $\frac{1}{2}(SP \times QT)$ . Given the relations  $d = QR$  and  $A = \frac{1}{2}(SP \times QT)$ , our force formula in Eq. (5) is equivalent to Newton's formula in Eq. (6).

Note that Eq. (6) measures the relative value of the force at a particular point in the orbit. The dimension of Newton's force measure  $QR/(SP \times QT)^2$  is  $1/(\text{length})^3$ . Like Newton, we will measure force in the purely geometric units of an inverse volume.<sup>5</sup> These units are perfectly fine because like Newton, we are only interested in comparing the force values at different points in the orbit. In what follows, the symbol  $F$  will denote the force measure  $QR/(SP \times QT)^2$ .

The force formula in Eq. (6) is the hallmark of Newton's geometric (Euclidean) approach to dynamics. The formula is not confined to celestial motion. The formula is valid for any kind of motion due to a centripetal force—any force directed toward a fixed point (force center).

## Newton's Recipe

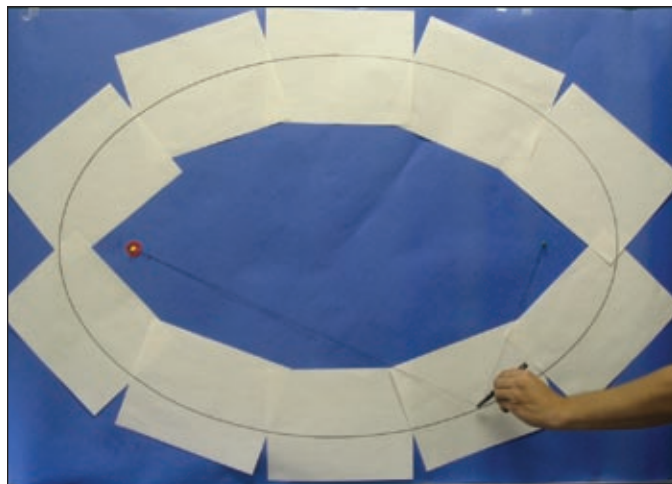
Given only two ingredients—the *shape of the orbit* and the *center of the force*—“Newton's Recipe” allows one to calculate the relative force at any orbital point. The recipe consists of the following steps:

1. **The inertial path:** Draw the tangent line to the orbit curve at the point P where the force is to be calculated.
2. **The future point:** Locate any future point Q on the orbit that is close<sup>6</sup> to the initial point P.
3. **The deviation line:** Draw the line segment from Q to R, where R is a point on the tangent, such that QR (line of deviation) is parallel to SP (line of force).
4. **The time line:** Draw the line segment from Q to T, where T is a point on the radial line SP, such that QT (height of “time triangle”) is perpendicular to SP (base of triangle).
5. **The force measure:** Measure the shape parameters QR, SP, and QT, and calculate the force measure  $QR/(SP \times QT)^2$ .
6. **The calculus limit:** Repeat steps two to five for several future points Q around P to obtain several force measures. Take the limit  $Q \rightarrow P$  of the sequence of force measures to find the exact value of the force measure at P.<sup>7</sup>

## Orbital Mechanics Laboratory

In our Orb Lab, students solve the *Kepler Problem* in three steps: (1) construct an elliptical orbit, (2) measure the force  $F$  at several points  $r$  on the orbit, (3) analyze the variation of  $F$  with  $r$  to find the law of force.

To construct an orbit, a team of about 10 students draws a large ellipse. Two tacks are pinned to a board. The ends of an inflexible string or thread are attached to the pins. Alternatively, a loop of string can be wrapped around the tacks. A pen held taut against the string traces out an elliptical curve. The length of the string determines the length of the major axis of the ellipse. The tacks represent the foci of the ellipse. One tack represents the Sun. Two different orbits are constructed: ellipse  $\alpha$  (major axis = 117.7 cm, eccentricity = 0.533) and ellipse  $\beta$  (major axis = 95.1 cm, eccentricity = 0.397).



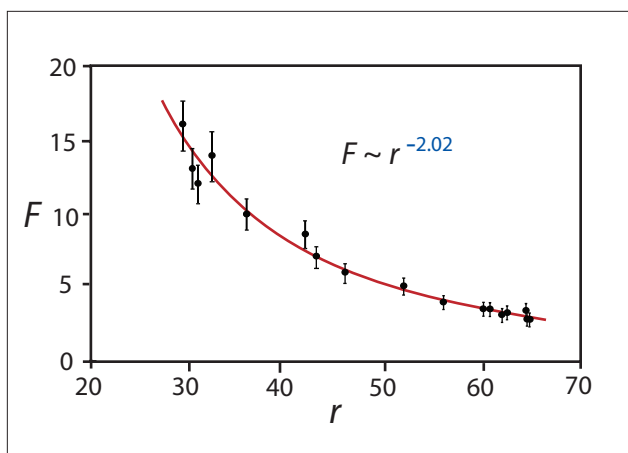
**Fig. 5.** The class constructs an elliptical orbit. Each student gets a small piece (arc) of the whole ellipse and measures the force responsible for the shape of his or her arc.

**Table II.** Values of the force  $F$  measured by a team of students at nine different radii  $r$  along their elliptical orbit. The team uncovers a simple pattern in the data:  $F = 1.23/r^2$ .<sup>12</sup>

$r$ (m)	$F$ ( $m^{-3}$ )
0.324	14.0
0.359	10.0
0.419	8.60
0.460	6.00
0.560	4.00
0.607	3.66
0.625	3.42
0.644	3.46
0.647	2.80

A photograph of an elliptical orbit partitioned into orbital arcs is shown in Fig. 5. Dividing the whole orbit into small pieces (parabolic arcs) breaks the celestial problem into manageable parts—projectile motion problems. On each arc, a point P representing the position of the planet is marked. The radial line connecting the Sun S and each P is drawn. The radial distance  $r = SP$  is written next to each P.

Each team member gets his or her own orbital arc to analyze. Using Newton's Recipe, each team member is responsible for measuring the value of the force  $F$  acting on the planet at his or her particular value of the Sun-planet distance  $r$ . The quest of the whole team is to find the force law—the continuous



**Fig. 6. Force law discovered by the class based on measuring the force  $F$  at 17 points  $r$  on the elliptical orbit.**

function  $F(r)$  that gives the values of the force  $F$  at all points  $r$  in the orbit. To find  $F(r)$  each student must share his or her value of  $F$  at one point with the whole class. We have students go to the blackboard and enter their pair of numbers ( $r$ ,  $F$ ) into a data table. Such a table is replicated in Table II.

Students analyze the class data to find a relationship between  $F$  and  $r$ . There is no guarantee that a simple theoretical function  $F(r)$  describes the experimental data. A belief in the simplicity (beauty) of nature or being acquainted with the law of gravity prompts students to try fitting the data with a power law. Using a graphing program, students plot  $F$  versus  $r$ , fit the points with the function  $F \sim 1/r^n$ , and find the best-fit value of the exponent  $n$ . An  $F$ -versus- $r$  graph is displayed in Fig. 6.

The results are quite accurate. Based on measuring  $F$  at only eight or nine points  $r$  along the ellipse, teams discovered force laws ranging from  $F \sim 1/r^{1.99}$  to  $F \sim 1/r^{2.14}$ . By merging their data, teams found the force law  $F \sim 1/r^{2.06}$  for ellipse  $\alpha$  and  $F \sim 1/r^{2.02}$  for ellipse  $\beta$ . Large classes can discover precise laws. Students are convinced that inverse-squared forces cause ellipse-shaped orbits.

### Non-celestial Force Laws: $r^{-5}$ , $r^{-3}$ , $r^1$

Newton's Recipe can be applied readily to other orbit problems. Although the Kepler problem is the relevant problem for celestial orbits (ellipses with Sun at focus), solving other orbit problems illustrates the power and versatility of the geometric theory. As an

experimental test of Newton's theoretical results, we have measured the force laws for the three non-celestial orbits listed in Table I (circle, spiral, ellipse with Sun at center). Our measured values (-4.92, -2.78, +0.93) of the force-law exponents are consistent with Newton's theoretical values (-5, -3, +1).

## Conclusion

Physicists are in the business of finding force laws. Newton's Recipe allows students to practice the art of discovering force laws for themselves. In the Orb Lab, students come face to face with the Kepler problem and are exposed to Newton's geometric formula. Drawing tangents and measuring deviations epitomize Newton's first and second laws. The experiment is low tech (string, tacks, ruler). The physics is basic ( $d = \frac{1}{2}at^2$ ,  $F = ma$ ,  $t \propto A$ ). The results are powerful (elliptical orbit  $\Rightarrow 1/r^2$  force). As a bonus, students receive hands-on lessons in elliptical geometry, infinitesimal calculus, and collaborative research (working as a team to find a law of nature).

## References

1. Isaac Newton, *The Principia, Mathematical Principles of Natural Philosophy*, a new translation by I. Bernard Cohen and Anne Whitman (University of California, Berkeley, 1999).
2. J. Bruce Brackenridge, *The Key to Newton's Dynamics* (University of California, Berkeley, 1995).
3. The general solution of  $F = ma$  for any force  $F$  is  $\mathbf{r} - \mathbf{r}_0 = \mathbf{v}_0 t + (\mathbf{F}_0/2m)t^2$ , where the interval of time is small enough for the force to be approximately constant ( $t \approx 0$ ,  $F \approx F_0$ ). Using Newton's notation, this vector equation of displacements is  $PQ = PR + RQ$ . The orbital displacement is  $PQ = \mathbf{r} - \mathbf{r}_0$ . The inertial displacement due to  $\mathbf{v}_0$  alone is  $PR = \mathbf{v}_0 t$ . The falling displacement due to  $\mathbf{F}_0$  alone is  $RQ = (\mathbf{F}_0/2m)t^2$ .
4. Newton's force measure,  $QR/(SP \times QT)^2$ , is introduced in Proposition 6, Book 1 of the *Principia* (Ref. 1, p. 453). Newton derived theoretical force functions  $F(r)$  by calculating  $QR/QT^2$  as a function of  $SP = r$  using Euclid's propositions. We derive "experimental" force functions by measuring  $QR$ ,  $SP$ , and  $QT$  using a ruler.
5. The inverse ratio  $(SP \times QT)^2/QR$  has units of volume. Indeed, Newton associates centripetal force with a hypothetical solid. In Proposition 6, Corollary 1 (Ref. 1, p. 454), Newton writes "... the centripetal force will

be inversely as the solid ( $SP^2 \times QT^2$ )/QR, provided that the magnitude of the solid is always taken as that which it has ultimately when the points P and Q come together.”

6. What does “close” mean? In theory, the force formula is exact only for infinitesimal deviations. In our experiment, we assume that a deviation is “infinitesimal” if the length of the deviation QR is less than 10% of the Sun-planet distance SP. Given the size of the orbits drawn in our class ( $30 \text{ cm} < SP < 90 \text{ cm}$ ), we suggest to the students that they choose future points Q around P so that the deviations QR are about 4 cm, 3 cm, 2 cm, and 1 cm. Measuring smaller distances with a ruler involve larger relative errors.
7. How do you take the “calculus limit?” If you are in the calculus regime of infinitesimal deviations (approximated by  $QR < 0.1SP$ ) and parabolic arcs, then the values of the force measures in the force sequence should be roughly constant or slowly approaching a limiting value. We have the students estimate the limit by simply looking at the values, noticing a trend, and performing a qualitative extrapolation (or average).

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