The Capital Structure of a Firm Under Rate of Return Regulation: Durability and the Yield Curve

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Abstract

I present a generalized dynamic model of firm behavior under rate of return regulation. The modeled firm has access to multiple types of capital which are substitutes (imperfect or perfect) in production. These capital inputs are differentiated based on durability and heterogeneous marginal effects on the firm's total cost of capital. This approach is kept general but is motivated by the stylized shape of the bond yield curve, wherein high durability (longer lived) assets command a higher required return on investment (higher bond yield). The results indicate that a regulated firm (relative to an unregulated firm) will over or under-invest in specific assets depending on their durability and the size of the assets marginal effect on the cost of capital relative to the regulated rate of return.

Two specific sources of distortion in the capital structure are identified. The "yield curve" effect pushes the firm further (relative to the unconstrained case) towards assets with a low marginal contribution to the cost of capital, thus reducing the firm's average cost of capital. The "duration" effect pushes the firm towards longer lived assets as a means to inflate the steady state capital stock. This is similar to (yet distinct from) the classic Averch and Johnson (1962) result.

Keywords Rate-of-Return ; Cost-of-Capital ; Depreciation ; Yield Curve

JEL codes L51, D21, D92

1 Introduction

A simple, standard and powerful result from the neo-classical model of firm behavior is the consistency between the input decisions of a profit-maximizing firm and the input decisions of a cost-minimizing firm. A consequence of the rate of return model of economic regulation is that it inadvertently ties a regulated firm's profits directly to its input decisions, invalidating this equivalence. Under a binding rate of return constraint, quantity and price choices do not affect firm profits in the same way as in an unregulated market. In this article I show that a firm subject to rate of return regulation can influence its profits in two ways: i) changing the margin between its permitted rate of return and cost of capital by investing in capital assets which lower its average cost of capital and ii) varying the total capital stock on which this margin is earned by over-investing in high durability assets which increases its steady-state capital stock.

Despite the emergence of incentive-based regulatory mechanisms (chiefly price caps) over the past half century, the rate of return methodology remains widely used. As indicated by Blank and Mayo (2009), price cap schemes, regarded as a prominent alternative to rate of return regulation, amount to a special case of the rate of return model with an institutionalized regulatory lag. At a primal level the standard rate of return framework undergirds most practical price regulation.[1]

The implicit objective of the rate of return model of economic regulation (and most economic regulation in general) is to reduce or eliminate the dead-weight loss associated with the exercise of market power. The intuition behind the rate of return approach is to impose a constraint on behavior intended to force a firm to set a price approximating its average cost. Through this mechanism a firm which would otherwise exercise market power is forced to mimic the behavior of a firm acting in a more competitive environment insofar as pricing and output are concerned. The rate of return constraint is essentially a constraint on the firm's revenue net of variable costs. The relative success of imposing the constraint (measured by the reduction in dead-weight loss) depends on two elements and the interaction between them. The first is the accuracy of the

[1] Another common alternative to standard rate of return regulation is the use of Negotiated Settlements between producers and consumers. In practice however, settlement outcomes are often based on, or are a modification of, the existing rate of return framework they are replacing. See Fellows (2011)

constraint (i.e. how close the regulated 'fair' rate of return is to the firm's actual cost of capital). The second is the degree of distortion in the firm's cost function caused by the imposition of the constraint. I take the accuracy of the rate-of-return constraint as given, and focus on the distortion of the firm's cost function implied by changes in the firm's input demand functions. The focus here is on the regulated firm's departure from the cost minimizing input mix, not on the overall reduction in dead-weight loss effected by imposing a rate of return constraint.

I construct and solve a model of firm behavior to identify the input demand functions of a profit-maximizing firm subject to a rate of return constraint. The model is dynamic in continuous time and includes an exogenous rate of return constraint and heterogeneous capital inputs differentiated by durability. Incorporating dynamics into the model allows for capital inputs to affect not just current production but also future costs and revenues. Less durable assets draw down the value of the capital stock faster, leading to a lower steady state capital stock. Changes in the average durability (and associated depreciation rate) of the asset stock also affects the principle repayment schedule and the perceived risk of the firm. This in turn may ultimately affect the firm's average cost of capital.

I identify two distorting effects that arise from the imposition of a rate of return constraint. The "duration" effect pushes the firm towards longer lived assets as a means to inflate the steady state capital stock. Under an assumption of imperfect regulation this will lead to increased profits as the larger capital stock will inflate the firm's revenue cap proportionally more than its costs. The "yield curve" effect pushes the firm further (relative to the unconstrained case) towards short-lived capital inputs, thus lowering its risk profile and by extension its average cost of capital. This has the effect of widening the margin between the regulated rate of return and the firm's average cost of capital.[2]

The remainder of the article is laid out as follows: Section 2 elaborates on the motivation for the modeling excercise conducted here, referencing relevant literature on the rate of return model, the capital durability choice faced by the firms and the resulting effect on capital costs. Section 3 describes the construction of the model and compares the resulting constrained and

[2]The referenced reduction in average cost of capital does not imply a reduction in overall capital cost, as the total stock of capital may or may not increase as a result of this shift.

unconstrained equilibrium. Section 4 concludes.

2 Further Motivation and Existing Literature

Starting with Averch and Johnson's (1962) seminal work there has been a substantial volume of analytical and empirical analysis on the subject of input distortions under rate of return constraint.[3] The model developed by Averch and Johnson indicates that a firm subject to a binding but imperfect rate of return constraint (where the regulated rate of return exceeds the firm's true average cost of capital) will choose a higher capital-labor ratio relative to an unconstrained firm for any output level.[4] This result is commonly referred to as the Averch and Johnson (or AJ) effect.

Katz (1983), Caputo and Partovi (2002) and others[5] have used various techniques to analyze the original AJ-style model. These authors show that the AJ effect cannot be established for a generalized version of the model due to the un-characterized sensitivity of the regulatory constraint's shadow value with respect to the choice variables. The most common ancillary assumption imposed to overcome this limitation is some variation on imposing a revenue function which generates isoquants concave in capital and labor. This assumption is often criticized because anecdotally we observe limited scope for capital-labor substitution in regulated industries.

The existing literature following from Averch and Johnson (1962) has generally focused on the predicted capital-labor distortion. However, the setup of the AJ model to include a practial rate of return constraint is much more important than its commonly understood result of capital-labour distortion. Averch and Johnson's model setup acknowledges the imperfect nature of a practical rate of return constraint in that the regulated rate of return is exogenously specified insofar as the firm's input decisions are concerned.

The concept of a "cost of capital" is abstract and not directly observable by the regulator. This implies that the regulator must use an estimate of the firm's cost of capital to set the

[3]Google Scholar indicates approximately 2452 citations of the original Averch and Johnson (1962) article, over 1000 of which are in the last decade alone. I therefore limit myself to a selection of noteworthy citations.

[4]The logic is straightforward; a regulated firm faces a revenue cap based largely on the product of a regulated rate of return and capital stock. Increasing capital inputs relative to labor inputs increases the revenue cap by a greater amount than costs and therefore leads to higher profits relative to the cost minimizing case.

[5]Takayama (1969); Pressman and Caron (1971, 1973); Jorgenson (1972); Hodiri and Takayama (1973)

regulated rate of return. In practice, this estimate is often based on a general base case and may be indexed to the yield on a government bond with a risk adjustment.[6]

Unfortunately, despite the insightful and intuitive realization by Averch and Johnson, their use of a static model fails to produce several important implications of this assumption. Their model also only admits a single capital and a single labor input which means that the indicated distortion (the AJ effect) is reliant on the aforementioned questionable assumption of capital labor substitutability. Applying the important intuitive concept of imperfect regulation to a static model with only two inputs severely limits its exploitation and this has evidently led current developments in regulatory theory to become increasingly dismissive of the AJ approach.

The model presented below, with dynamic optimization by firms and an explicit treatment of capital asset durability, generates new results and insights on the effect of an imperfect regulatory constraint on inter-capital substitution patterns with respect to asset durability. Placing the assumption of imperfect regulation in a dynamic model, the derived results do not rely on an assumption of capital-labor substitution. Instead, the focus is on inter-capital substitution in the revenue function. By this, I mean that the firm is assumed able to substitute between heterogeneous capital inputs differentiated by durability.[7]

This model could be viewed as something of a reconciliation between the work of Averch and Johnson (1962) who used a static model to examine input distortions under an imperfect rate of return constraint and Rogerson (1992) who used a dynamic model to investigate depreciation under a perfect rate of return constraint.

The most significant departure between the model I will present here and the aproaches of Averch and Johnson (1962) and Rogerson (1992) is the consideration for hetereogenous capital inputs differentiated by durability. This consideration is justified by two key insights. The first is that, firms are able to and in fact do substitute between capital inputs with varying degrees of durability. The second is that investments in assets with different degrees of durability may differently affect the firm's average cost of capital.

[6]For an example of this in practice, see: National Energy Board of Canada (1995).

[7]Elaborating on the capital labor substitution issue. The model developed below can accommodate, but does not rely on, and assumption of Leontief production between Labor and capital inputs. See footnote 16 for a technical description in the context of the model presented here.

The first insight has been explored in the neo-classical investment literature with considerable effort devoted to identifying the determinants of a firm's choice of capital-asset durability. Starting with Awerbach (1979, 1983), Abel (1981) and Gibbons (1984) and continuing with more recent work by Cohen and Hassett (1999) and Goolsbee (2004) the focus has been on how different corporate tax regimes effect investment decisions in durable assets. In this literature the firm is generally allowed to choose a single capital asset class (with a single associated durability/depreciation-rate) and the desired investment level. These models make an implicit assumption that assets with different durabilities are perfectly substitutable in the firm's production/revenue function. I draw on this literature but allow for a general form of inter-asset substitution (accommodating an assumption of imperfect substitutability in the firm's revenue function) between capital assets with different durabilities.

There are many ways a firm can substitute between low-durability and high-durability capital inputs. For example, a firm could choose between additional machinery to increase capacity on the factory floor and additional office equipment to better manage existing physical capacity. Either of these investments would increase overall productive capacity. Because low-durability office equipment is depreciated at a much faster rate than high-durability machinery, this choice amounts to substituting between inputs with a low and high depreciation rate.

Another example is the substitution between pipeline diameter and compression in the natural gas transmission industry. Given the physics of compressible fluid flow, pressure and pipe diameter produce a classic convex isoquant in the production of throughput (measured as cubic feet per day).[8] Allowing for maintenance costs, natural gas pipeline (the physical pipe in the ground) is treated as having an indefinite physical life and is therefore depreciated over 40 years, the longest period available under the general accepted accounting principles (GAAP). By com-

[8]Schroeder (2010) provides an equation governing one dimensional, compressible fluid flow, derived in part from the Fanning friction equation. Using Schroeder's generalized equation as a base, the fluid flow equation can be simplified as: $Q = D^{2.5} \cdot \sqrt{(P_{inlet}^2 - P_{outlet}^2)} \cdot \Theta$, where Q indicates cubic feet per day, P_{inlet} is the inlet pressure (PSIA), P_{outlet} is the outlet pressure (PSIA) and Θ is a function of other parameters including the length of the pipeline and the specific gravity of the gas being transported. Holding the pipe outlet pressure (and other parameters) constant, the marginal rate of technical substitution between inlet pressure and diameter is calculated as: $MRTS_{P_{inlet},D} = 2.5\frac{P_I^2 - P_O^2}{D \cdot P_I}$. which indicates a convex isoquant for any level of output Q. Whereas the associated isocost contour is unlikely to exhibit the classic linear shape, the associated optimum (or optima) would certainly be interior and exhibit marginal trade-offs.

parison compressor stations exhibit an average service life of between 20 and 25 years, with the specific service life (and corresponding depreciation period) heavily influenced by the projected gas load and run-times of the stations. Higher compression results in accelerated wear on the compressor stations and by extension a reduction in the expected service life and depreciation period of the asset.[9] Here again, there is a choice between investment in the durable pipeline and less durable compressor stations.[10]

The almost universal repair or replace decision faced by firms also represents a tradeoff in terms of durability of assets. The decision to replace implies that the firm pays off it's associated liabilities and allows the current asset to fall off the books faster adding a new asset to the capital stock, essentially resetting the depreciation clock. Conversely a decision to continue repairing existing capital assets allows the long-lived asset to continue depreciating at a slower rate. Continuing to repair existing assets would likely add smaller and shorter-lived assets to the capital stock (in the form of replacement components and/or repair materials), with the net effect being a lower average capital stock than if assets are more regularly replaced.[11]

Examples of the repair or replace decision are common and are faced by any mature firm employing durable capital. One example detailed by Anderson (2005) is the decision faced by nuclear power generators in timing the replacement of components of their steam generators. Anderson finds that the standard prospective time-line for replacing key components of the steam generator is between 19 and 25 years demonstrating considerable freedom in the repair/replace decision.

Despite the heterogeneity of capital assets with respect to durability a common and convenient convention adopted in the regulatory literature is to associate all capital assets with a single cost of capital unrelated to durability or depreciation rate. The conventionally recognized

[9]See: Westcoast Energy Inc's (2003) Transmission Depreciation Study for an example of practical discussion of these issues in the context of a regulated pipeline.

[10]The example here refers to substitutions made on a single physical pipeline, but it is simple to extend the analysis to the case of building a second parallel line running through the same compressor stations. This is referred to "looping" in the industry taxonomy.

[11]Unlike the previous two examples, the repair or replace decision is greatly impacted by the distinction between physical depreciation and financial depreciation. Repairing an asset (i.e., regular maintenance), influences the physical depreciation on an input by acting to preserve its productive capacity. However this change in physical depreciation may or may not be reflected in the financial depreciation rate determined by the GAAP. I comment further on this distinction in the model construction section below.

"cost of capital" is itself short for the "weighted average cost of capital" where the weighting is across a firm's debt and equity (rather than across assets of different durabilities). Buranabunyut and Peoples (2012, p.186) assert that the input decisions of the firm are based on shadow prices rather than actual prices. Following their assertion I abstract from the firm's financing decisions (choice of debt/equity ratio and bond duration) assuming that the regulated firm will internalize the *required* financial market return as a shadow or opportunity cost of purchasing any specific asset.[12] I explicitly consider the required return to specific *assets* included in the firm's capital stock.

Awerbuch (1995) and Dew-Becker (2012) suggest that assets with different associated risks and durations should be and in fact are discounted by firms at different rates. Awerbuch (1995) illustrates these effects in the context of a cost-benefit analysis for a firm choosing between capital projects with varying degrees of risk. He indicates that a firm conducting an appropriate cost-benefit analysis should internalize the effect a risky investment will have on its overall cost of capital.

Dew-Becker (2012) shows evidence of a similarly defined action on the part of firms. His empirical analysis of U.S. industries indicates a strong relationship between changes in the shape of the yield curve (i.e. the relative cost of long-term debt) and the investment decisions of firms with respect to long-lived (more durable) and short-lived (less durable) assets. As expected, an increase in the relative cost of long-term debt leads to a shift towards investment in short-lived assets. Any model which applies a single exogenous cost of capital would completely ignore these potentially important effects on investment decisions.

[12]The reasoning here is essentially the same as in the classic "Invariance Proposition" developed by Modigliani and Miller (1961). In broad terms, I appeal directly to the "Invariance Proposition" insofar as capital *financing* decisions are concerned. It is important to note that although the original Modigliani and Miller version of the invariance proposition holds when applied to financing decisions, the application of the invariance proposition to the choice of depreciation rate (or the choice between capital asset inputs with different deemed depreciation rates) used in calculating the regulated revenue stream (as in Schmalensee (1989)) does not hold. Changing the financing decisions between equity and debt does nothing to the present value of the income stream (the relevant decision is simply how to distribute this stream). However, changing the regulated depreciation rate (or the choice of capital inputs with different deemed depreciation rates) does have a direct effect on the present value of the revenue stream due to the regulatory constraint and/or any productivity implications.

Spiegel (1997) notes that the vast majority of rate of return literature implicitly assumes full equity financing (no debt). Some readers may wish to continue applying this assumption to the model below. Such interpretation should not materially harm any of the intuition or implications of the model as long as the reader accepts the general premise that each asset in the rate base can be associated with a distinct individual cost of capital potentially related to the asset's durability.

Even though the explicit risk treatment of asset-specific discounting is convincing, I restrict my examination of capital cost effects to the duration relationship identified by Dew-Becker (2012). In the model below I explicitly treat each heterogeneous capital input (differentiated by durability) as having potentially distinct marginal effects on the total capital cost.

The use of heterogeneous distinct marginal effects on the total capital cost is easily reconciled with the convention of assuming a single cost of capital. New capital investments can be considered to have a "direct" effect and an "indirect effect" on the firm's total capital cost. Considering that the total capital cost is defined as the product of the cost of capital and the capital stock; the direct effect on capital cost comes from the change in the value of the capital stock (consistent with the bulk of existing literature which treats capital as homogeneous). The indirect effect, currently ignored, is the result of a change in the value of the single "cost of capital" brought about by a change in the risk profile or repayment schedule of existing capital stock. Given this description, the costs-of-capital can be aggregated into a weighted average representing the conventional cost of capital, maintaining consistency between the two interpretations. Appendix A provides a derivation showing this consistency. I reject the common convention of referencing a single cost of capital, choosing instead to refer to *costs*-of-capital in order to simplify the model construction and discussion of results.

Despite evidence indicating that long-term assets carry a higher proportional capital cost (or positive infra-marginal effect on the average cost of capital), the model construction admits arbitrary pairs of depreciation rates and costs-of-capital for an asset. This maintains generality and avoids the need for a critical dependence on this assumption. Additional insight is gained by assuming that the yield curve relationship holds for physical assets of different durations (as illustrated by Table 1 and Figure 1 below), but the model's general implications (Propositions 1 through 4) do not rely on this assumption.

3 The Model

The model distinguishes between productive stocks of capital inputs and the associated liabilities (book values). The capital cost and revenue cap (under the rate of return constraint) are based on the book value (outstanding liability) of capital whereas incoming revenue is based on the productive value. To maintain the distinction between reductions in the productive values and associated liabilities, I introduce the taxonomy of depreciation and amortization. “Depreciation” is defined here as the periodic deterioration or reduction in the *productive capacity* of a physical asset whereas “Amortization” is defined as the rate of repayment of the liability associated with the physical asset (this can also be though of as a persistent reduction in the *book value* of an asset).

Distinct asset classes are represented in the model by the subscript i where $i \in N$. These classes are determined by an asset's a) productive quality (marginal product), b) depreciation rate, c) amortization rate and d) specific cost of capital. k_i represents the stock of an asset from asset class i whereas B_i represents the associated liability.

I place two minimally restrictive assumptions on the elements of set N. First, it is assumed that $i \in N$ if and only if the associated k_i is used in positive quantity by a firm when regulated and/or unregulated. This definition helps to avoid dealing with irrelevant capital inputs which generate corner solutions in both the constrained and unconstrained equilibrium. Second, I assume that there are at least two distinct elements in the set N.[13]

The modeled firm is assumed to borrow to finance any and all asset purchases. For a given asset class of type i a new investment is recorded as both an addition to the stock of physical value of capital k_i and as an addition to the firm's associated liability B_i. Since borrowing is assumed to directly offset asset purchases in any period t, the cost of new investments does not appear explicitly in the cash flow equation. Rather, the cost of investment shows up in the form of a principal repayment schedule wherein the stock of the firm's liabilities is repaid to the lenders over time along with a return on the currently outstanding liability. The net cash flow for an asset i at the margin is the marginal revenue product it generates less the associated amortization and interest payments.

[13]If there is only one element in the set N, then the model collapses to a something approximating a standard AJ model (with added dynamics, which become largely irrelevant since the firm has only one feasible choice for its average depreciation rate). This special case of the model makes no useful contribution to extant literature. I direct interested readers to Caputo and Partovi (2002), who provide a generalized and comprehensive examination of a the static AJ model.

Under these assumptions, the firm's net cash flow in any period can be given as:

$$Net\ Cash\ Flow = F(K_t, L_t) - W \cdot L_t - \sum_{i \in N} ((r_i + \alpha_i) \cdot B_{i,t}) \tag{1}$$

where the vector K_t is the set of capital inputs at time t ($k_{i,t}\ \forall i \in N$). W and L_t represent the standard wage and labor (at time t) input decision, respectively.[14] α_i is the amortization rate for asset class i and r_i is the specific cost of capital for asset class $i.F(K_t, L_t)$ is a continuous revenue function twice differentiable for each element of the set represented by K_t. The revenue function is assumed jointly concave in all capital inputs.[15,16]

The rate of return constraint is defined as:

$$\sum_{i \in N} (S \cdot B_{i,t}) \geq F(K_t, L_t) - WL_t - \sum_{i \in N} (\alpha_i \cdot B_{i,t}) \tag{2}$$

where S is an exogenous rate of return set by the regulator. The firm acts to maximize net cash flow defined by equation (1) subject to this constraint.

Closure of the model requires equations of motion for the two sets of state variables $k_{i,t}$ and $B_{i,t}$. For computational ease, and to abstract from cost overrun or revenue shortfall discussions, the equations of motion are specified in continuous time. In all of the equations of motion, a dot above a variable ($\dot{}$) indicates a time derivative. The set of equations of motion for $k_{i,t}$ are:

$$\dot{k}_{i,t} = I_{i,t} - \delta_i \cdot k_{i,t}\ \forall i \in N \tag{3}$$

where $I_{i,t}$ are units of new capital investment by the firm and δ_i is the depreciation rate for asset class i. By similar constructions the equations of motion for B_i are:

[14] I implicitly assume that W and L_t are a single scalar and variable, however the model is robust to including a vector of differentiated labor inputs and associated wage rates in which case W and L_t would be vectors.

[15] More specifically the function $F(K_t, L_t)$ is assumed to produce a negative definite Hessian Matrix, with elements; $\left\{ \frac{\partial F(K_t,L_t)}{\partial k_{i,t}} > 0\ ;\ \frac{\partial^2 F(K_t,L_t)}{\partial k_{i,t}^2} < 0 \right\} \forall i \in N$ and $\frac{\partial^2 F(K_t,L_t)}{\partial k_{i,t} \partial k_{i,t}} > 0\ \forall i \neq j \in N$

[16] As indicated above, the model is robust to a Leontief relationship between capital and labor. Formally, $F(K_t, L_t)$ can take the form: $F(K_t, L_t) = F\left(min\left\{L_t, G(K_t)\right\}\right)$ (where $G(K_t)$ is some function of the vector K_t) without loss of generality.

$$\dot{B}_i = I_{i,t} - \alpha_i \cdot B_{i,t} \; \forall i \in N \tag{4}$$

As indicated above, the model is robust to any arbitrary pairing of α_i and δ_i. It is possible that $\alpha_i = \delta_i$ for all or some of the elements $i \in N$, however; the two are kept distinct in order to accommodate situations wherein the principle repayment schedule associated with a liability does not equal the underlying depreciation on the asset purchased.

From equations (1) through (4) the firm's maximization program in continuous time is:

$$\underset{L_t, I_{i,t}}{\text{Max}} \int_0^T \left(\begin{array}{r} F(K_t, L_t) - WL_t - \sum_{i \in N} ((r_i + \alpha_i) \cdot B_{i,t}) \\ -\lambda \left(F(K_t, L_t) - WL_t - \sum_{i \in N} ((S + \alpha_i) \cdot B_{i,t}) \right) \end{array} \right) e^{-r_f t} dt$$

$$\begin{aligned} S.T. \;\; \dot{k}_{i,t} &= I_{i,t} - \delta_i \cdot k_{i,t} \;\; \forall i \in N \\ \dot{B}_i &= I_{i,t} - \alpha_i \cdot B_{i,t} \;\; \forall i \in N \\ k_{i,t} &\geq 0 \; \forall i \in N \\ I_{i,t} &\geq 0 \; \forall i \in N^S \end{aligned}$$

where the newly introduced variable r_f represents the discount rate taken as exogenous by the firm.[17] Two sets of non-negativity constraints are also included in the maximization program. The non-negativity constraint on k_i is intuitive while the additional non-negativity constraint on I_i incorporates the potential for "sunk capital" to enter the model. The new set notation N^S in the last non-negativity constraint represents the subset of asset classes for which investment is sunk ($N^S \subseteq N$). I allow for an infinite time/planning horizon for the firm such that $t \in [0, \infty)$.

From this point on I drop all time subscripts (t), to simplify notation. I solve for an equilibrium to the maximization program outlined above through the use of a Current Value Hamiltonian function. Construction of the Current Value Hamiltonian consists of multiplying the

[17] r_f is the firm's discount rate on cash flow. Because there is no explicitly modeled risk in the net cash flow equation, r_f is assumed to be exogenous through time and is treated as a fixed short-term risk-free rate of return.

objective function and it's constraints by the term $e^{r_f t}$ and then combining them using standard Lagrangian methods.[18] The Current Value Hamiltonian representing the above maximization program is given as:

$$\begin{aligned} H = F(K,L) - W \cdot L - \sum_{i \in N} ((r_i + \alpha_i) \cdot B_i) \\ -\lambda \left(F(K,L) - W \cdot L - \sum_{i \in N} ((S + \alpha_i) \cdot B_i) \right) \\ + \sum_{i \in N} q_i \left(I_i - \delta_i \cdot k_i \right) + \sum_{i \in N} \eta_i \left(I_i - \alpha_i \cdot B_i \right) \\ + \sum_{i \in N} \mu_i \cdot k_i + \sum_{i \in N^S} \gamma_i \cdot I_i \end{aligned} \tag{5}$$

where q_i and η_i are the costate variables for the productive value and liability of each capital stock i, and μ_i and γ_i are the Karush-Kuhn-Tucker multipliers associated with the non-negativity constraints on capital inputs and investment.[19,20]

The corresponding first order conditions for the firm's choice of L and I_i are given by equations (6) and (7):

$$\frac{\partial F}{\partial L} = W \tag{6}$$

$$q_i + \eta_i + \gamma_i = 0 \;\; \forall i \in N^S \tag{7a}$$

$$q_i + \eta_i = 0 \;\; \forall i \in N \not\subset N^S \tag{7b}$$

The first order condition for the firm's choice of labor is familiar and has the standard (marginal revenue product equals marginal cost) interpretation. The first order conditions for capital investments are not as easily interpreted. These conditions require that the co-state

[18] For more details see: Chiang (2000), section 8.2, pages 210-212.

[19] I ignore the non-negativity constraint on labor input, as it lends no useful insight into the model implications.

[20] Despite the revenue constraint, both the firm's output, and it's revenues are endogenous and jointly determined by the level of capital investment. Since the firm must produce output sufficient to generate the revenues it is allowed to collect under the rate-of-return constraint, there are no explicit minimum quality or quantity constraints in equation (5).

variables representing shadow values on productive capital and those representing shadow costs of associated liabilities must balance at the firm's optimum. In the case of sunk assets (equation (7a)) the additional presence of the Lagrange multiplier (γ_i) represents the shadow value of the constraint ($I_i \geq 0$) when it is binding.

Given the methodology used to construct the Current Value Hamiltonian, the costate variables (q_i and η_i) implicitly include the term $e^{r_f t}$. With this in mind the maximum principal conditions for the state variables k_i and B_i are :

$$(1-\lambda) \cdot \frac{\partial F}{\partial k_i} + \mu_i - q_i \cdot \delta_i = r_f q_i - \dot{q}_i \ \ \forall i \in N \tag{8}$$

$$\lambda S - r_i - (1-\lambda) \cdot \alpha_i - \eta_i \cdot \alpha_i = r_f \eta_i - \dot{\eta}_i \ \ \forall i \in N \tag{9}$$

In a standard neo-classical investment model, the absence of convex adjustment costs would imply a direct jump to the steady state. Despite the absence of such adjustment costs here the modeled firm cannot normally jump to the optimal levels for k_i and B_i simultaneously.This is due to the limitation that the firm has only one control variable (I_i) for every two state variables (k_i,B_i). Nevertheless, a stable, optimal steady state equilibrium exists. Appendix B provides additional details on the dynamics of the system outside of the steady state as well as illustrating that $I_i = \delta_i \widehat{k}_i$ where $\widehat{k}_i$ is the optimal steady state value of k_i.

Imposing the feasible steady state condition such that $\dot{q}_i = 0$ and $\dot{\eta}_i = 0$ the maximum principal conditions can be rewritten as in equations (10) and (11):

$$(1-\lambda) \cdot \frac{\partial F}{\partial k_i} + \mu_i = (r_f + \delta_i) \cdot q_i \ \ \forall i \in N \tag{10}$$

$$\lambda \cdot S - r_i - (1-\lambda) \cdot \alpha_i = (r_f + \alpha_i) \cdot \eta_i \ \ \forall i \in N \tag{11}$$

Due to the non-negativity constraint on k_i combined with the steady state condition that $I_i = \delta_i \widehat{k}_i$ it is evident that at the steady state $I_i \geq 0$ which implies that $\gamma_i = 0 \ \forall i \in N^S$. As such, equations (7a) and (7b) can be rewritten in the steady state as:

$$q_i + \eta_i = 0 \;\; \forall i \in N \tag{12}$$

Combining equations (10) and (11) with equations (12), implies the following equilibrium conditions for each asset class i:

$$\widehat{\frac{\partial F}{\partial k_i}} = \left(\frac{r_f + \delta_i}{r_f + \alpha_i}\right) \cdot \left(\frac{r_i - \lambda S}{1-\lambda} + \alpha_i\right) - \left(\frac{1}{1-\lambda}\right)\mu_i \quad \forall i \in N \tag{13}$$

The hat ($\hat{}$) over the left hand side of equation (13) denotes that this is the equilibrium value for the partial derivative (marginal revenue product) for the constrained version of the model. This equation implicitly defines the firm's choice of capital stock for each asset class i as a function of the model's exogenous parameters. Taken together, the conditions for each asset class i defined by equation (13) and the first order condition for labor determine the firm's profit-maximizing input choices.

The unconstrained equivalent of equation (13) is:

$$\frac{\partial F^*}{\partial k_i} = \left(\frac{r_f + \delta_i}{r_f + \alpha_i}\right)(r_i + \alpha_i) - \mu_i \;\; \forall i \in N \tag{14}$$

where the star (*) on the left hand side of the equation indicates that this is the unconstrained profit maximizing value of the partial derivative. This equation is derived either by removing the constraint from equation (5) and re-solving the equilibrium, or more directly by setting $\lambda = 0$ in equation (13).

As indicated above, theoretical work following from the AJ effect indicates that a characterization of that effect is dependent on the size of the shadow value on the constraint.[21] Despite the differences between the model constructed here, and the more traditional static AJ formulation, the size of the Lagrange multiplier λ continues to play a vital role. Lemma 1 characterizes the feasable range of values for λ:

Lemma 1 (Feasible Values for the Lagrange Multiplier)**.** *For the model defined by the current*

[21] See Caputo and Partovi (2002) for a review and reconciliation of the analytical work characterizing the shadow value on the rate of return constraint for a static Averch and Johnson (1962) style model.

value Hamiltonian in equation 5; if an equilibrium exists for the constrained version of the model, then the equilibrium value of the Lagrange multiplier satisfies $\lambda \in (0, 1)$

See appendix C for the associated proof.[22]

I define a new set $M \subset N$ where $i \in M$ if and only if $\widehat{k_i} > 0$ *and* $k_i^* > 0$. Use of this set along with the result in lemma 1 allows further characterization of the existence of corner solutions in the constrainted and unconstrained equilibria. Lemma 2 formalizes this characterization.

Lemma 2 (Corner Solutions in the Constrained and Unconstrained Model)**.** *For the model defined by the current value Hamiltonian in equation 5; if both a constrained and unconstrained equilibrium exist then* $\forall i \in N$*;*

- *if* $S > r_i: \quad \widehat{k_i} = 0 \quad \Longrightarrow \quad k_i^* = 0.$
- *if* $S < r_i: \quad k_i^* = 0 \quad \Longrightarrow \quad \widehat{k_i} = 0.$
- *if* $S = r_i: \quad \widehat{k_i} = 0 \quad \Leftrightarrow \quad k_i^* = 0.$

See appendix C for the associated proof.

Lemma 2 indicates that for capital with a low cost of capital ($r_i < S$), a corner solution in the constrained model implies a corner solution in the unconstrained model. That is, there is no low cost of capital input used in the unconstrained case that is not used in the constrained case. Lemma 2 also indicates that for capital with a high cost of capital ($k_i > S$), a corner solution in the unconstrained model implies a corner solution in the constrained model. That is, there is no high cost of capital input used in the constrained case that is not used in the unconstrained case.

Given the statetments in lemmas 1 and 2 the regulatory distortion can be characterized by comparing equations (13) and (14). Proposition 1 details this distortion at a basic level.

Proposition 1 (Changes in Volumes of Capital Inputs from rate of return)**.** *From equations* (13) *and* (14)*; if both a constrained and unconstrained equilibrium exist for the maximization*

[22] Together, equations (7) and (10) and lemma 1 imply that $q_i \geq 0$ and $\eta_i \leq 0$ $\forall i \in N$. This is intuitive as capital used in production should have a positive effect on the future profit stream whereas the liability of that capital stock should have a negative effect on the future profit stream.

problem represented by equation 5 then in response to a binding rate of return constraint a profit maximizing firm will:

- *increase its use of any capital asset ($i \in N$) with a cost of capital below the regulated rate of return.*
- *maintain its use of any capital asset ($i \in N$) with a cost of capital equal the regulated rate of return.*
- *decrease its use of any capital asset ($i \in N$) with a cost of capital above the regulated rate of return.*

See appendix C.1 for the associated proof.

Proposition 1 indicates that, for capital inputs with a cost of capital below the regulated rate, the outcome under a dynamic model is consistent with the over-capitalization indicated in the original AJ model. Whats more, because the model allows for multiple capital inputs, the equilibrium defined by equations (6) and (13) are able to admit positive values of inputs for which the cost of capital is above the regulated rate of return ($r_i > S$).

Arguably the most prominent question examined in the analytical and empirical work following from the original model presented in Averch and Johnson (1962) is that of over-capitalization. Given the inclusion of heterogeneous capital inputs in this model, a discussion of the traditional "over-capitalization" result requires consideration for the relative distortion of investment in capital from each asset class.

Given the general case presented here it is impossible to fully characterize the relative total size of the capital stock for a regulated and unregulated firm. A sufficient (but not necessary) condition for aggregate over-capitalization is that $S > max\,(r_i)\ \forall i \in N$. This condition implies, via proposition 1, that: $\widehat{k}_i > k_i^* \ \forall i \in N \quad \Longrightarrow \quad \sum_{i \in N} \left(\widehat{k}_i\right) > \sum_{i \in N} (k_i^*)$.

Continuing with the application of proposition 1, if $max\,\{r_{i \in N}\} > S > min\,\{r_{i \in N}\}$ there will be both downward biased and upward biased elements of the capital stock. In this case, characterizing the bias in the aggregate capital stock requires complete information on the size of the terms $(r_i - S)\ \forall i \in N$ as well as a specific functional form for the firm's revenue function

$F(K,L)$. With these assumptions in place, the same logic used in the proof accompanying proposition 1 can be employed to characterize the bias in the aggregate capital stock.

Absent the inclusion of heterogeneous capital, $r > S$ would imply negative profits for a regulated firm employing any positive quantity of capital. Allowing for heterogeneous capital with different costs of capital the constrained firm may still find it profitable to employ a positive quantity of k_i when $r_i > S$ as long as the condition $S \geq \left(\sum \left(r_i \cdot \widehat{B}_i\right) / \sum \left(\widehat{B}_i\right)\right)$ holds.[23] Thus, the last bulleted clause of proposition 1 is included not only for completeness, but also as a potential outcome for some subset of the set of heterogeneous capital inputs N for which the individual costs of capital are above the regulated rate of return.

Given the existence of multiple capital inputs we can further characterize the distortions by examining the marginal rate of technical substitution (MRTS) at equilibrium for each pair of inputs in the constrained and unconstrained cases.

In the remainder of this section I will use $i \in N$ and $j \in N$ to denote two different asset classes ($i \neq j$).

Equation (15) is derived as a simple ratio of equations (10) evaluated for asset classes i and j. It defines the equilibrium value of the marginal rate of technical substitution between any two asset classes.[24]

$$\widehat{MRTS}_{i,j} = \widehat{\left(\frac{\partial k_j}{\partial k_i}\right)} = \frac{(r_f + \delta_i)(r_f + \alpha_j)}{(r_f + \delta_j)(r_f + \alpha_i)} \left(\frac{r_i + \alpha_i - \lambda(S + \alpha_i)}{r_j + \alpha_j - \lambda(S + \alpha_i)}\right) \tag{15}$$

Either via a similar derivation from equation (14) or by setting $\lambda = 0$ in equation (15), the equilibrium rate of substitution in the unconstrained case can be characterized as:

$$MRTS^*_{i,j} = \left(\frac{\partial k_j}{\partial k_i}\right)^* = \frac{(r_f + \delta_i)(r_f + \alpha_j)}{(r_f + \delta_j)(r_f + \alpha_i)} \left(\frac{r_i + \alpha_i}{r_j + \alpha_j}\right) \tag{16}$$

A comparison of equations (15) and (16) will characterize differences in the marginal rate

[23]Recall that: $\widehat{B}_i = \frac{\delta_i}{\alpha_i}\widehat{k}_i$ and that $\widehat{k}_i$ is a function of S. Thus the inequality referenced will produce a threshold value of S, below which the firm will choose $\widehat{k}_i = 0$ and cease to produce in the optimal steady state.

[24]Equation 15 draws from the derivatives of a revenue function $F(\ )$, rather than directly from a production function. However, for a single product firm, the definitions are identical. $\frac{\left(\frac{\delta F}{\delta k_i}\right)}{\left(\frac{\delta F}{\delta k_j}\right)} = \frac{\left(\frac{\delta F}{\delta Q}\right)\left(\frac{\delta Q}{\delta k_i}\right)}{\left(\frac{\delta F}{\delta Q}\right)\left(\frac{\delta Q}{\delta k_j}\right)} = \frac{\left(\frac{\delta Q}{\delta k_i}\right)}{\left(\frac{\delta Q}{\delta k_j}\right)} = MRTS_{i,j}$

of technical substitution between heterogeneous capital inputs and by extension the volume of these inputs in the constrained and unconstrained cases. This comparison requires a more complete characterization of the endogenous variable λ, provided in lemma 3 and its associated proof:

Lemma 3 (Stronger conditions on λ). *For the model defined by the current value Hamiltonian in equation 5; if an equilibrium exhibiting positive profits exists for the constrained version of the model, then the equilibrium value of the Lagrange multiplier satisfies $\lambda \in (0, \overline{\lambda})$, where $\overline{\lambda}$ is the largest value that satisfies $\left(\overline{\lambda} \leq \frac{r_i+\alpha_i}{S+\alpha_i}\right) \forall i \in M$.*

See appendix C for the associated proof.

Imposing a simplifying set of assumptions on α_i and α_j as well as r_i and r_j and comparing the second term on the right hand side of equations (15) and (16) illustrates two types of bias on the firm's input decisions resulting from a binding rate of return constraint. These are detailed in propositions 2 and 3.

Proposition 2 (cost of capital Bias). *From equations* (15) *and* (16), *assuming equal amortization rates ($\alpha_i = \alpha_j$);and differences in individual costs-of-capital ($r_i > r_j$); If an equilibrium to the maximization problem represented by equation 5 exists, a binding rate of return constraint will lead a firm to* **increase** *its use of lower-cost capital inputs (k_j) relative to its use of higher-cost capital inputs (k_i) for all $i, j \in N$, compared to an unconstrained firm.*

See appendix C.2 for the associated proof.

Without a full examination of the proof behind proposition 2 its meaning is easily misunderstood. In general we expect that a profit maximizing firm facing the type of well behaved revenue function described above will reduce its use of an input when faced with an increase in that input's costs (substituting away from high cost inputs towards low cost inputs). Proposition 2 is not a redundant restatement of this behavior.

In the unconstrained case, the rate of return earned on an asset is endogenous to the model. The firm makes input decisions to maximize total profits, and the rate of return can then be calculated by dividing these profits over the total capital stock. The profit maximizing input decisions in the unconstrained case are based on marginal revenues and marginal costs only.

In the constrained case, the rate of return is exogenously set by the regulator and is no longer endogenous to the model. Faced with a rate of return constraint the firm is also concerned with how input decisions act to tighten or relax this constraint. This distorts the standard marginal revenue/marginal cost equalization and leads to the behavoir described by proposition 2 where the firm shifts too far (relative to the unconstrained/cost-minimizing case) towards low cost capital.

Taken together with the earlier assertion that longer duration assets are associated with higher costs-of-capital, proposition 2 implies that a firm will shift towards lower-cost short-term capital. As foreshadowed above, I dub this distortion as the "yield curve effect." However, in isolation the yield curve effect ignores the effect that differing amortization rates have on the steady state liability associated with the productive capital stock and by extension the revenue cap imposed under rate of return regulation.

Varying the amortization rates while holding the costs of capital constant between asset classes illustrates another distortionary effect.

Proposition 3 (The Duration Effect)**.** *From equations* (15) *and* (16)*, assuming differences in the amortization rates of capital inputs* ($\alpha_i < \alpha_j$)*; and equal costs of capital below the regulated rate* ($S > r_i = r_j$)*: If an equilibrium to the maximization problem represented by equation 5 exists; a binding rate of return constraint will lead a firm to* **increase** *its use of longer-lived capital inputs* (k_i) *relative to its use of short-lived capital inputs* (k_j) *for all* $i, j \in N$ *when compared to an unconstrained firm.*

See appendix C.3 for the associated proof.

Because it is the liability (or book value) of the firm's capital assets that determines the firm's regulated revenue requirement (and by extension the regulated revenue cap and profits), the allocation of investment between assets with different amortization rates will affect the firm's profits. Longer lived assets (those with lower depreciation rates) are preferred because they remain on the books for longer, increasing the steady state capital stock. This increase in capital stock relaxes the rate of return constraint and increases the revenue cap. An unregulated firm is only concerned with amortization insofar as it affects the costs associated with an investment.

This difference is the source of the distortion described by proposition 3.

Through propositions 1 and 3 the model indicates that a regulated firm will over-invest in any capital with an individual cost of capital below the regulated rate, and that this over-investment will be more pronounced for capital with lower amortization rates if all costs of capital are equal.

Propositions 2 and 3 are based on an inconsistent set of assumptions insofar as differences in the costs-of-capital (r_i ,r_j) and amortization rates (α_i, α_j) are concerned. Proposition 1 continues to hold, although allowing for differences in both the costs-of-capital and the amortization rates (such that $r_i \neq r_j$ and $\alpha_i \neq \alpha_i$) significantly complicates the analysis of the effects identified by propositions 2 and 3.

As stated above, an observation of the yield curve indicates that lower amortization rates will in general imply higher costs of capital. Imposing assumptions consistent with this assertion, the duration effect and yield curve effect will drive the distortions in investment in opposite directions insofar as asset duration is concerned. For a constrained firm, the duration effect makes long-lived capital more attractive (relative to short lived capital) whereas the yield curve effect makes short-lived capital more attractive (relative to long-lived capital). Proposition 4 characterizes the condition required for dominance of either effect given assumptions consistent with the assertion that longer lived capital carries a higher cost of capital.

Proposition 4 (The Dominant Effect)**.** *From equations* (15) *and* (16)*: If an equilibrium to the maximization problem represented by equation 5 exists:*

- *If* $\left((r_j + \alpha_j) - (r_i + \alpha_i) < \frac{(r_i\alpha_j - r_j\alpha_i)}{S}\right)$*, then:*$\left(\frac{\widehat{k_i}}{\widehat{k_j}} < \frac{k_i^*}{k_j^*}\right)$*; (Duration Effect Dominates)*
- *If* $\left((r_j + \alpha_j) - (r_i + \alpha_i) > \frac{(r_i\alpha_j - r_j\alpha_i)}{S}\right)$*, then:*$\left(\frac{\widehat{k_i}}{\widehat{k_j}} > \frac{k_i^*}{k_j^*}\right)$*; (Yield Curve Effect Dominates)*
- *If* $\left((r_j + \alpha_j) - (r_i + \alpha_i) = \frac{(r_i\alpha_j - r_j\alpha_i)}{S}\right)$*, then:*$\left(\frac{\widehat{k_i}}{\widehat{k_j}} = \frac{k_i^*}{k_j^*}\right)$*; (The Effects Exactly Offset)*

See appendix C.4 for the associated proof.[25]

Thus, the direction of the bias with respect to asset duration is ambiguous but identifiable for any pair of asset classes given information on the regulated rate of return, amortization rates

[25] Note that, implicitly by its statement, and following from the proof in appendix C.4 Proposition 2 does not hold if $k_i^* + k_j^* = 0$.This is intuitive because we cannot compare proportional increases in the use of two variables if they are used in zero quantities in the unconstrained case.

and costs-of-capital. Table 1 exhibits a set of numerical examples to indicate the general pattern of the dominant effect with respect to exogenous parameter values.[26]

Table 1: Numerical Examples for Proposition 4

Case	Parameters			Condition from Proposition 4			Dominant Effect
	r_2	r_{10}	S	Left Side		Right Side	
Base	2.5%	5%	6%	−0.375	=	−0.375	N/A
$+\Delta S$	2.5%	5%	**7%**	−0.375	<	−0.321	Duration
$-\Delta S$	2.5%	5%	**5%**	−0.375	>	−0.450	Yield Curve
$+\Delta r_{10}$	2.5%	**6%**	6%	−0.365	>	−0.458	Yield Curve
$-\Delta r_{10}$	2.5%	**4%**	6%	−0.385	<	−0.292	Duration
$+\Delta r_2$	**3.5%**	5%	6%	−0.385	<	−0.358	Duration
$-\Delta r_2$	**1.5%**	5%	6%	−0.365	>	−0.392	Yield Curve

$\alpha_2 = 50\%$, $\alpha_{10} = 10\%$

The second and third rows ($+\Delta S$ and $-\Delta S$) of table 1 show that (holding all other parameters constant) an increase in the exogenous regulated rate of return S will weaken the yield curve effect relative to the duration effect, whereas a reduction in S has the opposite effect.

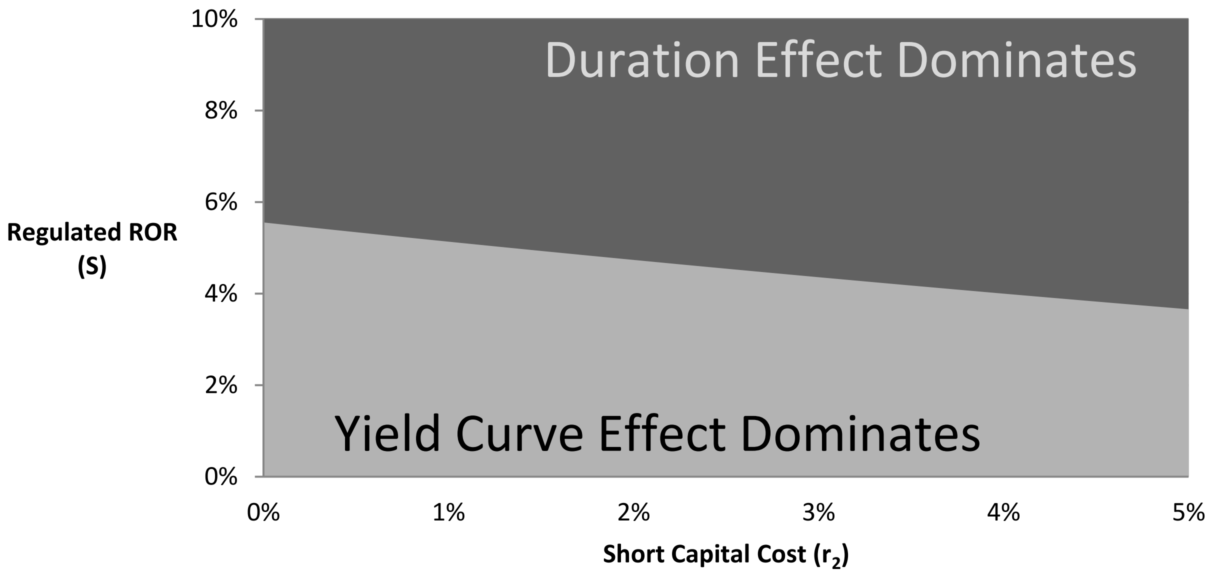
The fourth and fifth rows ($+\Delta r_{10}$ and $-\Delta r_{10}$) of table 1 show that, holding all other parameters constant, a increase in the required return on long term capital (ten year in the case of table 1) will strengthen the yield curve effect relative to the duration effect. Likewise, a reduction in r_{10} has the opposite effect.

Finally, the last two rows ($+\Delta r_1$ and $-\Delta r_1$) show that, holding all other parameters constant, an increase in the required return on short term capital (two year in the case of table 1) will weaken the yield curve effect relative to the duration effect. Again, the opposite is true, with a reduction in r_1 strengthening the amortization effect relative to the yield curve effect.

Fixing r_{10} at 5% and varying r_2 and S (as an example) the relationship described in proposition 4 is exposited graphically in Figure 1. As in the numerical examples in Table 1, increasing S weakens the yield curve effect moving the firm into the range wherein the duration effect dominates. As illustrated the duration effect is very likely to dominate if the regulated rate of return is above the ten year bond yield ($r_{10} = 5\%$). Figure 1 also shows that an increase in the

[26] The specific values used in the table are not directly based on empirical data; however, the values are generally consistent with the range of estimated or observed corollaries found in the existing literature.

Figure 1: Dominant Effect Areas



Holding Constant: $\alpha_2 = 50\%$ $\alpha_{10} = 10\%$ $r_{10} = 5\%$

short term bond yield rate (r_2) also weakens the yield curve effect.

It is interesting to note that the specific characteristics of the revenue function and how k_i and k_j enter into it have no bearing on the parameters that determine which effect is dominant. Although the specifics of the revenue function have bearing on the severity or magnitude of the two effects, the direction of any distortion is determined solely by the parameters listed in proposition 4.

The magnitude of a distortion caused by either the yield curve effect or duration effect will be greater for inputs with a high degree of substitutability. If two capital inputs enter linearly in the production function then the constraint may potentially cause a shift between two corner solutions ($k_i^* > 0; k_j^* = 0$ to $\widehat{k_i} = 0; \widehat{k_j} > 0$).

4 Conclusion

The model presented above serves as a step towards understanding the effect a rate of return constraint has on a firm's input decisions when accounting for dynamic effects and heterogeneous capital. The results presented extend the standard AJ interpretation of over-capitalization and provide important new implications related to amortization and cost of capital considerations. The assumptions underlying this model are arguably more realistic than the static workhorse AJ model and the results are more directly applicable to real world regulation (where historical investment and amortization decisions determine much of a firm's revenue requirement and ultimately the profits and regulated price).

The model implications are intriguing both in their similarity to and departures from the implications of current theory. Proposition 1 implies that an effect similar to Averch-Johnson style over-capitalization carries over to the dynamic model for any asset with a cost of capital lower than the regulated rate of return, however; the model also presents novel contributions in the area of input decisions between capital assets with different amortization rates and associated capital cost characteristics. The characterized differences in the firm's input demand functions with and without the rate of return constraint also indicate that under reasonable assumptions the regulated firm is not cost minimizing. Therefore, under the assumption of an imperfect rate of return constraint (where the regulated 'fair' rate is above the firm's true cost of capital) not only is the effective price above the firm's actual average cost, but the firm's actual average cost is above the cost minimizing average cost.

A Details on the Costs-of-Capital Approach

The standard convention in economics and finance is to refer to a single average cost of capital. The capital cost implicitly defined by equation (1) can be made consistent with this convention with only a slight modification. Consider that the total capital cost can be given as: $Capital\ Cost = r_a \cdot \sum (B_{i,t})$ where the new variable r_a is the commonly referenced weighted average cost of capital. The per unit change in total capital cost for a change in investment in a specific asset class i can then be calculated as:

$$\frac{d\left[Capital\ Cost\right]}{dB_{i,t}} = \underbrace{r_a}_{\text{direct}} + \underbrace{\frac{\partial r_a}{\partial B_{i,t}} \cdot \sum_{i \in N} (B_{i,t})}_{\text{indirect}}$$

It is the indirect effect in the above differential that is commonly ignored by restricting capital investments to a homogeneous cost of capital.

Imposing the form $r_a = \sum_{i \in N} (r_i \cdot B_{i,t}) \,/ \sum_{i \in N} (B_{i,t}))$, the total differential above can be simplified as:

$$\frac{d\,[Capital\ Cost]}{dB_i} = \underbrace{r_a}_{\text{direct}} + \underbrace{(r_i - r_a)}_{\text{indirect}} = r_i$$

consistent with the costs-of-capital interpretation.

B Dynamics of the System

I present here a sketch for the proof of dynamic stability (convergence to the steady state) for the optimal control problem described by the Hamiltonian given in equation (5). I begin this sketch with an illustration that the system is stable in a subspace comprised of the state variables. I then illustrate that, for each asset class ($i \in N$), the ratio of the liability (B_i) to the productive value (k_i) of capital is known. Finally I provide a simple phase diagram in k_i, B_i space illustrating that the system is stable around the optimal steady state equilibrium.

For each element in set N (the total set of distinct asset classes) there are 3 accompanying dynamic variables: $I_{i,t}$, $k_{i,t}$ and $B_{i,t}$. Consider the lower dimension $2N$ subset of the dynamic system composed of equations (3) and (4). The system can be written in the standard matrix ($\dot{X} = Ax + c$):

$$\begin{pmatrix} \dot{k}_1 \\ \dot{B}_1 \\ \dot{k}_2 \\ \dot{B}_2 \\ \vdots \\ \dot{k}_N \\ \dot{B}_N \end{pmatrix} = \begin{pmatrix} -\delta_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\delta_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\alpha_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\delta_N & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha_N \end{pmatrix} \begin{pmatrix} k_1 \\ B_1 \\ k_2 \\ B_2 \\ \vdots \\ k_N \\ B_N \end{pmatrix} + \begin{pmatrix} I_1 \\ I_1 \\ I_2 \\ I_2 \\ \vdots \\ I_N \\ I_N \end{pmatrix}$$

The set of eigenvalues for this subspace can be characterized by the set of values for Λ which satisfy[27]:

$$\prod_{i \in N} \left[(\delta_i + \Lambda) (\alpha_i + \Lambda) \right] = 0 \quad \Longrightarrow \quad \Lambda \in \{-\delta_i\} \cup \{-\alpha_i\}$$

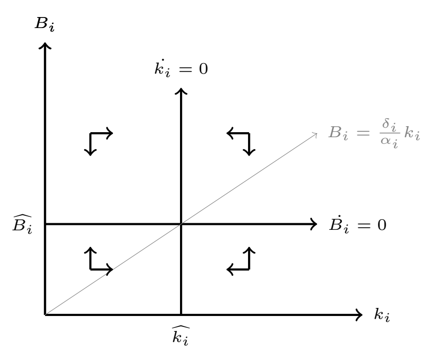
Thus the set of eigenvalues for the subspace dimension is uniformly negative. The system is therefore stable in both of the state variables for each asset class via the standard stability conditions. For any given level of the control variables I_i the state variables k_i and B_i will converge to a steady state.

The ratio of between k_i and B_i in the steady state is fixed and can be derived for any given level of investment. Consider a case in which investment in an asset class i is fixed through time at a value $\bar{I}$. In this situation, k_i and B_i will each converge to a steady state where:

$$\left.\begin{array}{l} \dot{k}_i = 0 \implies 0 = \bar{\imath} - \delta_i k_i \implies k_i = \dfrac{\bar{\imath}}{\delta_i} \\ \dot{B}_i = 0 \implies 0 = \bar{\imath} - \alpha_i B_i \implies B_i = \dfrac{\bar{\imath}}{\alpha_i} \end{array}\right\} \implies B_i = \frac{\delta_i}{\alpha_i} k_i$$

Thus, a feasible path to the steady state exists. Full characterization of this path implies that the firm will choose levels for the control variables $I_{i,t}$ which will ensure that all k_i and B_i converge to the optimal steady state $(\widehat{k_i}, \widehat{B_i})$ following the maximization program outlined in the main paper. At the optimal steady state $I_i = \delta_i \widehat{k_i} = \alpha_i \widehat{B_i} \implies \dot{I}_i = 0$.

Figure 2: Phase Diagram for B_i,k_i when $I_i = \delta_i \widehat{k_i} = \alpha_i \widehat{B_i}$



The phase diagram for the 2 dimensional subset k_i, B_i of the system around the steady state

[27]This equation is derived by setting $det\,(A - \Lambda \imath) = 0$ where A is the Jacobian matrix of the $2N$ system given above and $\imath$ is the identity matrix.

(i.e. when $I_i = \delta_i \widehat{k_i} = \alpha_i \widehat{B_i}$) is given in figure 2.

C Mathematical Appendix

Proof of Lemma 1. Arrow's sufficiency conditions for the existence of an equilibrium requires the Hamiltonian function to be jointly concave with respect to the state variables. Joint concavity of k_i and k_j requires that:

$$|b_1| = \begin{vmatrix} 0 & \frac{\partial H}{\partial k_i} \\ \frac{\partial H}{\partial k_i} & \frac{\partial^2 H}{\partial k_i^2} \end{vmatrix} = -\left(\frac{\partial H}{\partial k_i}\right)^2 \leq 0$$

(which is satisfied for any input k_i and function H) and

$$|b_2| = -(1-\lambda)\left[\left(\frac{\partial H}{\partial k_i}\right)^2 \frac{\partial^2 F}{\partial k_j^2} + \left(\frac{\partial H}{\partial k_j}\right)^2 \frac{\partial^2 F}{\partial k_i^2} - 2\frac{\partial H}{\partial k_i}\frac{\partial H}{\partial k_j}\frac{\partial^2 F}{\partial k_i \partial k_j}\right] \geq 0$$

Where b_1 and b_2 are the first and second principle minors of the bordered Hessian of (5). $\frac{\partial F}{\partial k_i} > 0\ \forall i \in N$ and the set of equations given in (10) imply: $sign\{\frac{\partial H}{\partial k_i}\} \equiv sign\{\frac{\partial H}{\partial k_j}\}\ \forall i,j \in N$. Thus: $\frac{\partial H}{\partial k_i} \cdot \frac{\partial H}{\partial k_j} > 0\ \forall i,j \in N$. Imposing $\left\{\frac{\partial^2 F}{\partial k_i^2} < 0\ ;\ \frac{\partial^2 F}{\partial k_{i,t} \partial k_{j,t}} \geq 0\right\}\ \forall\ i \in N$, the above relationship implies $\lambda < 1$. A binding rate of return constraint implies $\lambda > 0$ via the standard Karesh-Kuhn-Tucker conditions. Therefore $\lambda \in (0,1)$. □

Proof of Lemma 2. From equations (13) and (14):

$$\widehat{\mu_i} = max\left\{\left[\left(\frac{r_f+\delta_i}{r_f+\alpha_i}\right)\cdot\left(r_i+\alpha_i-\left(\frac{\lambda}{1-\lambda}\right)(S-r_i)\right) - \frac{\partial F}{\partial k_i}\bigg|_{k_i=0}\right](1-\lambda)\ ,\ 0\right\}$$
$$\mu_i^* = max\left\{\left[\left(\frac{r_f+\delta_i}{r_f+\alpha_i}\right)\cdot(r_i+\alpha_i) - \frac{\partial F}{\partial k_i}\bigg|_{k_i=0}\right]\ ,\ 0\right\}$$

From lemma 1: $\lambda \in (0,1)$. Thus:

- If $S > r_i$: $\widehat{\mu_i} > 0 \implies \mu_i^* > 0$, it follows that $\widehat{k_i} = 0 \implies k_i^* = 0$.
- If $S < r_i$: $\mu_i^* > 0 \implies \widehat{\mu_i} > 0$, it follows that $k_i^* = 0 \implies \widehat{k_i} = 0$.

- If $S = r_i : \mu_i^* > 0 \Leftrightarrow \widehat{\mu_i} > 0$, it follows that $\widehat{k_i} = 0 \Leftrightarrow k_i^* = 0$.

$\square$

Proof of Lemma 3. Via the definition of M: $k_i > 0 \quad \forall i \in M$. Imposing the condition $\left(\frac{\partial F}{\partial k_i} \geq 0 \;\; i \in M\right)$ equation (13) implies: $\lambda \leq \left(\frac{r_i+\alpha_i}{S+\alpha_i}\right) \;\; \forall i \in M$ if and only if $\lambda \in (0,1)$. From lemma 1: $\lambda \;\in (0,1)$. Define $\overline{\lambda}$ as the largest value that satisfies $\left(\overline{\lambda} \leq \frac{r_i+\alpha_i}{S+\alpha_i}\right) \;\; \forall i \in M$. Then $\lambda \in (0, \overline{\lambda})$. $\square$

C.1

Proof of Proposition 1. Subtracting equation (14) from equation (13) forms a set of equalities given by:

$$\widehat{\frac{\partial F}{\partial k_i}} - \frac{\partial F^*}{\partial k_i} = (r_i - S)\left(\frac{\lambda}{1-\lambda}\right) \;\; \forall i \in M$$

(recall that $\mu_i = 0 \;\; \forall i \in M$) Taking a linear approximation of the change on the left hand side of the above set of equalities, the set can be given in matrix notation as: $\mathcal{H}K_\Delta = \mathcal{C}$ where $\mathcal{H}$ is the MxM Hessian matrix associated with the capital inputs $i \in M$ of the revenue function $(F(K,L))$, K_Δ is an $Mx1$ vector composed of elements $\widehat{k_i} - k_i^*$ and $\mathcal{C}$ is a vector of values $(r_i - S)\left(\frac{\lambda}{1-\lambda}\right)$. As a Hessian matrix, $\mathcal{H}$ is a symmetric Hermitian matrix. The function $F(K,L)$ is assumed jointly concave in all capital inputs such that $\mathcal{H}$ is a negative definite matrix and $x^T\mathcal{H}x < 0$ for any real vector x. It follows that: $K_\Delta^T \mathcal{H} K_\Delta = K_\Delta^T \mathcal{C} < 0$

Note that the row and column orderings in the Hessian matrix $\mathcal{H}$ are arbitrary so long as symmetry is maintained. As a negative definite matrix, all upper left symmetric sub-matrices of $\mathcal{H}$ are themselves negative definite. Therefore the above equality holds for any sub-matrix of $\mathcal{H}$ and the inequality set implies that:

$$\left(\widehat{k_i} - k_i^*\right) \cdot \left((r_i - S)\left(\frac{\lambda}{1-\lambda}\right)\right) < 0 \;\; \forall i \in M$$

If an equilibrium exists, then via lemma 1: $\lambda \in (0,1)$, thus:

$$\left.\begin{cases} r_i < S & \implies (r_i - S)\left(\dfrac{\lambda}{1-\lambda}\right) < 0 & \implies \widehat{k}_i > k_i^* \\ r_i > S & \implies (r_i - S)\left(\dfrac{\lambda}{1-\lambda}\right) > 0 & \implies \widehat{k}_i < k_i^* \\ r_i = S & \implies (r_i - S)\left(\dfrac{\lambda}{1-\lambda}\right) = 0 & \implies \widehat{k}_i = k_i^* \end{cases}\right\} \forall i \in M$$

To this point the proof is established for capital inputs $i \in M$. For the set $\{i \in N | i \notin M\}$; following directly from lemma 2 and the definition of the set N:

$$\left.\begin{cases} r_i < S & \implies \widehat{k}_i > k_i^* = 0 \\ r_i > S & \implies k_i^* > \widehat{k}_i = 0 \end{cases}\right\} \forall i \in N | i \notin M$$

Thus the set of inequalities holds for all $i \in N$:

$$\left.\begin{cases} r_i < S & \implies \widehat{k}_i > k_i^* \\ r_i > S & \implies \widehat{k}_i < k_i^* \\ r_i = S & \implies \widehat{k}_i = k_i^* \end{cases}\right\} \forall i \in N$$

$\square$

C.2

Proof of Proposition 2 .

$$\left.\begin{aligned} r_i &> r_j \\ \alpha_i = \alpha_j &= \alpha \end{aligned}\right\} \implies \quad Sr_j + \alpha r_j < Sr_i + \alpha r_i$$

$$\implies \quad Sr_j + S\alpha - r_i r_j - \alpha r_i < Sr_i + S\alpha - r_i r_j - \alpha r_j$$

$$\implies \quad (S - r_i)(r_j + \alpha) < (S - r_j)(r_i + \alpha)$$

From lemma 3, if an equilibrium exists: $\lambda \in (0, \overline{\lambda})$.

$$\left.\begin{aligned} (S - r_i)(r_j + \alpha) < (S - r_j)(r_i + \alpha) \\ \lambda \in (0, \overline{\lambda}) \end{aligned}\right\} \implies -\lambda(S - r_i)(r_j + \alpha) > -\lambda(S - r_j)(r_i + \alpha)$$

$$\implies \frac{(r_i + \alpha)(1 - \lambda) - \lambda(S - r_i)}{(r_j + \alpha)(1 - \lambda) - \lambda(S - r_j)} > \frac{r_i + \alpha}{r_j + \alpha}$$

$$\implies \frac{r_i + \alpha_i - \lambda(S + \alpha_i)}{r_j + \alpha_j - \lambda(S + \alpha_j)} > \frac{r_i + \alpha_i}{r_j + \alpha_j}$$

Substituting equations (15) and (16) for the left and right hand side of the above inequality respectively, the inequality can be rewritten as: $\left\{\widehat{\left(\frac{\partial k_j}{\partial k_i}\right)} > \left(\frac{\partial k_j}{\partial k_i}\right)^*\right\} \forall i \neq j \in M$ (recall that $\widehat{\mu_i} = \mu_i^* = 0$ if and only if $i \in M$). Imposing declining marginal revenue product for both inputs:

$$\left.\begin{aligned} \widehat{\left(\frac{\partial k_j}{\partial k_i}\right)} > \left(\frac{\partial k_j}{\partial k_i}\right)^* \\ \frac{\partial F}{\partial k_i} > 0 \; \forall i, j \in N \\ \frac{\partial^2 F}{\partial k_i^2} < 0 \; \forall i, j \in N \end{aligned}\right\} \implies \left\{\frac{\widehat{k_i}}{\widehat{k_j}} < \frac{k_i^*}{k_j^*}\right\} \forall i \neq j \in M$$

Following directly from lemma 2 and the definition of the set N:

$$\left\{\frac{k_j^*}{k_i^*} < \frac{\widehat{k_j}}{\widehat{k_i}}\right\} \forall \{j \in N | j \notin M\} \;,\; \forall i \in M$$

although the fraction $\frac{k_j^*}{k_i^*}$ is undefined $\forall \{i, j \in N | i, j \notin M\}$ Therefore, $S > r_i > r_j$ and $\alpha_i = \alpha_j \implies \left\{\frac{k_j^*}{k_i^*} < \frac{\widehat{k_j}}{\widehat{k_i}}\right\} \forall i \neq j \in N$ if and only if $k_i^* + k_j^* > 0$. □

C.3

Proof of Proposition 3.

$$\left.\begin{aligned} r = r_i = r_j \\ \alpha_i < \alpha_j \end{aligned}\right\} \implies (S - r)(r + \alpha_j) > (S - r)(r + \alpha_i)$$

From lemma 3, if an equilibrium exists $\lambda \in (0, \overline{\lambda})$:

$$\left.\begin{aligned} (S-r)(r+\alpha_j) > (S-r)(r+\alpha_i) \\ \lambda \in (0,\overline{\lambda}) \end{aligned}\right\} \implies -\lambda(S-r)(r+\alpha_j) < -\lambda(S-r)(r+\alpha_i)$$

$$\implies \frac{(r+\alpha_i)(1-\lambda)-\lambda(S-r)}{(r+\alpha_j)(1-\lambda)-\lambda(S-r)} < \frac{r+\alpha_i}{r+\alpha_j}$$

$$\implies \frac{r_i+\alpha_i-\lambda(S+\alpha_i)}{r_j+\alpha_j-\lambda(S+\alpha_j)} < \frac{r_i+\alpha_i}{r_j+\alpha_j}$$

Substituting equations (15) and (16) for the left and right hand side of the above inequality respectively, the inequality can be rewritten as: $\left\{\widehat{\left(\frac{\partial k_j}{\partial k_i}\right)} < \left(\frac{\partial k_j}{\partial k_i}\right)^*\right\} \forall i \neq j \in M$. Imposing declining marginal revenue product for both inputs:

$$\left.\begin{aligned} \widehat{\left(\frac{\partial k_j}{\partial k_i}\right)} < \left(\frac{\partial k_j}{\partial k_i}\right)^* \\ \frac{\partial F}{\partial k_i} > 0 \; \forall i,j \in N \\ \frac{\partial^2 F}{\partial k_i^2} < 0 \; \forall i,j \in N \end{aligned}\right\} \implies \left\{\frac{\widehat{k_i}}{\widehat{k_j}} > \frac{k_i^*}{k_j^*}\right\} \forall i \neq j \in M$$

From lemma 2 if $S = r_i$ then $i \in M$ by definition. By extension, the only set of elements i that satisfies $S = r_i$ and $i \in N | i \notin M$ is the null set. Therefore: $r = r_i = r_j$ and $\alpha_i < \alpha_j \implies \left\{\frac{\widehat{k_i}}{\widehat{k_j}} > \frac{k_i^*}{k_j^*}\right\} \forall i \neq j \in N$. □

C.4

Proof of proposition 4.

$$(r_j+\alpha_j)-(r_i+\alpha_i) < \frac{r_i\alpha_j - r_j\alpha_i}{S} \implies Sr_j + S\alpha_j + \alpha_i r_j + \alpha_i\alpha_j < Sr_i + S\alpha_i + r_i\alpha_j + \alpha_i\alpha_j$$

$$\implies (S+\alpha_i)(r_j+\alpha_j) < (S+\alpha_j)(r_i+\alpha_i)$$

From lemma 1, if an equilibrium exists $\lambda \in (0, \overline{\lambda})$:

$$\left.\begin{aligned}(S+\alpha_i)(r_j+\alpha_j) < (S+\alpha_j)(r_i+\alpha_i)\\ \lambda \in (0,\overline{\lambda})\end{aligned}\right\}$$

$$\implies \quad (r_i+\alpha_i)(r_j+\alpha_j) - \lambda(S+\alpha_i)(r_j+\alpha_j) > (r_i+\alpha_i)(r_j+\alpha_j) - \lambda(S+\alpha_j)(r_i+\alpha_i)$$

$$\implies \quad \frac{r_i+\alpha_i-\lambda(S+\alpha_i)}{r_j+\alpha_j-\lambda(S+\alpha_j)} > \frac{r_i+\alpha_i}{r_j+\alpha_j}$$

Substituting equations (15) and (16) for the left and right hand side of the above inequality respectively, the inequality can be rewritten as: $\left\{\widehat{\left(\frac{\partial k_j}{\partial k_i}\right)} > \left(\frac{\partial k_j}{\partial k_i}\right)^*\right\} \forall i \neq j \in M$. Imposing declining marginal revenue product for both inputs:

$$\left.\begin{aligned}\widehat{\left(\frac{\partial k_j}{\partial k_i}\right)} > \left(\frac{\partial k_j}{\partial k_i}\right)^*\\ \frac{\partial F}{\partial k_i} > 0 \; \forall i,j \in N\\ \frac{\partial^2 F}{\partial k_i^2} < 0 \; \forall i,j \in N\end{aligned}\right\} \quad \implies \quad \left\{\frac{\widehat{k_i}}{\widehat{k_j}} < \frac{k_i^*}{k_j^*}\right\} \forall i \neq j \in M$$

The alternative effect; $(r_j+\alpha_j) - (r_i+\alpha_i) < (r_i\alpha_j - r_j\alpha_i)/S \implies \frac{\widehat{k_i}}{\widehat{k_j}} > \frac{k_i^*}{k_j^*} \;\; \forall i \neq j \in M$ follows from a parallel proof wherein the direction of the inequities is reversed. The null effect; $(r_j+\alpha_j) - (r_i+\alpha_i) = (r_i\alpha_j - r_j\alpha_i)/S \implies \frac{\widehat{k_i}}{\widehat{k_j}} = \frac{k_i^*}{k_j^*} \;\; \forall i \neq j \in M$ follows from a parallel proof wherein an equality is substituted for the inequalities. Via lemma 2, using the same logic as in the preceding proofs, these results can be generalized to elements $i \neq j \in N$ unless $k_i^* = k_j^* = 0$. □

References

[1] Abel A.B., *Taxes, Inflation and the Durability of Capital* Journal of Political Economy, 89, no. 3, 548-560 (1981)

[2] Anderson, T. S. *Diablo canyon power plant: steam generator repair/replacement cost/benefit analysis.* In Proceedings of the 2005 Crystal Ball User Conference. (2005)

[3] Averch, H. and Johnson, LL., *Behaviour of the Firm Under Regulatory Constraint.* The American Economic Review, 52, no. 5, 1052-1069. (1962)

[4] Auerbach, A. J., *Wealth maximization and the cost of capital.* The Quarterly Journal of Economics 93, no. 3, 433-446. (1979)

[5] Auerbach, A.J., *Taxes, Corporate Financial Policy and the Cost of Capital* Journal of Economic Literature, 21, 905-940 (1983)

[6] Awerbuch, S., *Market-based IRP: It's easy!!!* The Electricity Journal, 8, no. 3, 50-67. (1995)

[7] Bougheas, S. and Worrall, T., *Cost padding in regulated monopolies* Journal of Industrial Organization, 30, no. 4, 331-341. (2012)

[8] Buranabunyut, N. and Peoples, J., *An empirical analysis of incentive regulation and the allocation of inputs in the US telecommunications industry* Journal of Regulatory Economics, 41, no. 2, 181-200. (2012)

[9] Caputo, M. R., and Partovi, M. H., *Reexamination of the AJ effect* Economics Bulletin, 12, no. 10, 1-9. (2002)

[10] Chiang, A. C. Elements of dynamic optimization. Illinois: Waveland Press Inc. (2000)

[11] Cohen, D. and Hassett, K.A., *Inflation, Taxes, and the Durability of Capital* Division of Research & Statistics and Monetary Affairs, Federal Reserve Board, (1997).

[12] Dew-Becker, I., *Investment and the Cost of Capital in the Cross-Section: The Term Spread Predicts the Duration of Investment* mimeo, Harvard University, (2012)

[13] Fellows, G. K., *Negotiated Settlements with a cost of service backstop: the consequences for depreciation* Energy Policy, 39, no. 3, 1505-1513. (2011)

[14] Gibbons, J.C., *The Optimal Durability of Fixed Capital When Demand is Uncertain* The Journal of Business, 57, no. 3, 389-403. (1984)

[15] Goolsbee, A., *Taxes and the Quality of Capital* Journal of Public economics, 88, no. 3, 519-543. (2004)

[16] El-Hodiri, M.A., and Takayama, A., *Behavior of the Firm under Regulatory Constraint: Clarifications* American Economic Review, 63, no. 1, 235-237. (1973)

[17] Jorgenson, D. W., *Investment Behavior and the Production Function* The Bell Journal of Economic and Management Science, 220-251. (1972)

[18] Katz, Michael L. *A general analysis of the Averch-Johnson effect.* Economics Letters 11, no. 3, 279-283. (1983)

[19] Modigliani, F., and Miller, M. H., *The Cost of Capital, Corporation Finance and the Theory of Investment* American Economic Review, 48, no. 3, 261-297. (1958)

[20] National Energy Board of Canada, *RH-2-94, reasons for decisions, multi-pipeline (Cost of Capital),* March (1995) http://publications.gc.ca/collections/Collection/NE22-1-1995-1E.pdf

[21] Pressman I., and Carol, A., *Behaviour of the Firm under Regulatory Constraint: Note* American Economic Review, 61, no. 1, 210-212. (1971)

[22] Pressman I., and Carol, A., *Behaviour of the Firm under Regulatory Constraint: Reply* American Economic Review, 63, no. 1, (1973)

[23] Rogerson, W. P. *Optimal depreciation schedules for regulated utilities.* Journal of Regulatory Economics 4, no. 1, 5-33. (1992)

[24] Schmalensee, R. *An Expository Note on Depreciation and Profitability Under rate of return Regulation* Journal of Regulatory Economics, 1, no. 3, 293-298. (1989)

[25] Schroeder Jr, D. *A Tutorial on Pipe Flow Equation* PSIG Annual Meeting. 2010.

[26] Takayama, A., *Behavior of the Firm under Regulatory Constraint* American Economic Review, 59, no. 3, 255-260. (1969)

[27] Westcoast Energy Inc's *Transmission Depreciation Study* (2003)

https://www.neb-one.gc.ca/ll-eng/livelink.exe/fetch/2000/90465/92833/92844/586981/305270/310870/304770/A0J4S6_-_Tab_1_Depreciation_Study.pdf?nodeid=304786