PARTIAL FRACTIONS

Proper Rational Fraction:

If the degree of $f(x)$ is less than the degree of $g(x)$ then $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials, is called a proper rational fraction. Every proper rational fraction can be expressed (at least, theoretically) as a sum of similar fractions (partial fractions) whose denominators are of the form $(ax+b)^n$ and $(ax^2+bx+c)^n$, n being a postive integer. Four cases, depending upon the nature of the factors of the denominator, arise.

CASE 1: Distinct linear factors

To each linear factor $ax+b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be determined.

$$\text{Example: } \frac{3x}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} (\Rightarrow A = \frac{3}{2}, B = \frac{3}{2}).$$

CASE 2: Repeated linear factors

To each linear factor $ax+b$ occurring n times in the denominator, there corresponds a sum of n partial fractions of the form $\frac{A_1}{ax+b} + \frac{A}{(ax+b)^2} + \dots + \frac{A_n}{(ax+b)^n}$, where the A's are constants to be determined.

$$\text{Example: } \frac{3x+5}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} (\Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}, C = 4).$$

CASE 3: Distinct quadratic factors

To each irreducible quadratic factor ax^2+bx+c occurring once in the denominator, there corresponds a single partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.

$$\text{Example: } \frac{x^3+x^2+x+2}{x^4+3x^2+2} = \left(\frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)}\right) = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} \left(\Rightarrow \begin{array}{ll} A=0, & B=1, \\ C=1, & D=0. \end{array}\right)$$

CASE 4: Repeated quadratic factors

To each irreducible quadratic factor ax^2+bx+c occurring n times in the denominator, there corresponds a sum of n partial fractions of the form $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$, where the A's and B's are constants to be determined.

$$\text{Example: } \frac{x^5-x^4+4x^3+8x-4}{(x^2+2)^3} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}$$
$$(\Rightarrow A=1,\ B=-1,\ C=0,\ D=0,\ E=4,\ F=0).$$