UNIVERSITY OF CALGARY - FACULTY OF SCIENCE
Department of Mathematics and Statistics
FINAL EXAMINATION
Amat 307, L01 - L06, Fall 2005

Time 3 hours, NO AIDS, NO CALCULATORS

Surname	Given Names	Lec.	ID	Score/100

Examination Rules

1. Students late in arriving will not normally be admitted after one-half hour of the examination time has passed.

2. No candidate will be permitted to leave the examination room until one-half hour has elapsed after the opening of the examination, nor during the last 15 minutes of the examination. All candidates remaining during the last 15 minutes of the examination period must remain at their desks until their papers have been collected by an invigilator.

3. All enquiries and requests must be addressed to supervisors only.

4. **Candidates are strictly cautioned against:**

   (a) speaking to other candidates or communicating with them under any circumstances whatsoever;

   (b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;

   (c) making use of calculators and/or portable computing machines not authorized by the instructor;

   (d) leaving answer papers exposed to view;

   (e) attempting to read other student's examination papers.

   The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.

5. Candidates are requested to write answers on the front of the page, reserving the back for rough drafts. If you have insufficient room on the front, clearly indicate when you have work on the back page to be marked.

6. Discarded matter is to be struck out and not removed by mutilation of the examination answer book. Do not unstaple the pages of your exam.

7. Candidates are cautioned against writing in their answer book any matter extraneous to the actual answering of the question set.

8. The candidate is to write his/her name on each answer book as directed, as well as lecture number and student ID number.

9. A candidate must report to a supervisor before leaving the examination room. If you finish early, return the examination paper to the supervisor.

10. Cease writing promptly when the end-of-exam signal is given. Remain at your desk until the exam paper has been collected. Failure to comply with this regulation will be cause for rejection of an answer paper.

11. If a student becomes ill or receives word of domestic affliction during the course of an examination, he/she should report at once to the Supervisor, hand in the unfinished paper and request that it be cancelled. Thereafter, if illness is the cause, the student must go directly to the University Health Services so that any subsequent application for a deferred examination may be supported by a completed medical statement form. An application for Deferred Final Examination must be submitted to the Registrar by the date specified in the University Calendar.

    Should a student write an examination, hand in the paper for marking, and later report extenuating circumstances to support a request for cancellation of the paper and for another examination, such a request will be denied.

12. Smoking during examinations is strictly prohibited.

1. Calculate the following:

(a) the general solution of $ty' + 3y = t^2$ | 3 |

(b) the general solution of | 3 |
$(ye^{xy} + 2xy)dx + (xe^{xy} + x^2 + y)dy = 0$

(c) the equilibrium solutions of the autonomous equation $y' = (y-1)(y-5)(y-6)$ and classify them as stable, unstable, semistable. | 3 |

(d) the general solution of $y'' + 3y' + 2y = e^{-2t}$ 3

(e) the Wronskian $W(t)$ of the equation 3

$y'' + \dfrac{1}{t+1}\, y' + \cos(t)\; y = 0$, given that $W(0) = 3$.

(f) the general solution of equation $y^{(4)} + 8y^{(2)} + 16y = 0$. 3

(g) the coefficients a_n for the series expansion $e^{2x} = \sum_{n=0}^{\infty} a_n x^n$, and its radius of convergence.

3

(h) the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2 3^n}$; does this series converge at the point $x = -2$?

3

(i) the Laplace transform of $y(t) = u_2(t)(t-2)^3 e^t$, where $u_2(t) = \begin{cases} 0, & t < 2, \\ 1, & t \geq 2. \end{cases}$

3

(j) the inverse Laplace transform of $F(s) = \dfrac{2}{s^2+6s+9}$. | 3 | |

(k) the inverse Laplace transform of $F(s) = \dfrac{2s+1}{s^2-2s+2}$. | 3 | |

(l) The matrix $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ has eigenvectors | 3 | |

$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. What is the general solution for the homogeneous linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$?

(m) the solution of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ such that $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, where A is a constant matrix which has eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, associated with $\lambda = 1, 3$, respectively. | 3 | |

(n) the general solution of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ where $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ | 3 | |

(o) the general solution of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ where $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ | 3 | |

2. The population of mosquitoes in Calgary increases at a rate proportional to the current population, and in the absence of predators, the population doubles every seven days. Suppose there are initially 200,000 mosquitoes, and predators are introduced that eat 20,000 mosquitoes per day. Write down the differential equation modeling the population of mosquitoes and determine the population at any time t (measure time in days). | 6 | |

3. Solve the initial value problem $y' = \dfrac{t^2 + y^2}{ty}$, $y(1) = 4$. | 7 | |

4. Find the general solution of $y'' - 4y' + 5y = \sin(2t) + 8t^2$. | 7 | |

5. Find the general solution of the Euler equation | 7 | |

$$x^2 y'' - 2y = \frac{1}{x}$$

6. Find the first four terms in the series solution to the initial value problems $y'' - xy' - y = 0, y(0) = 2, y'(0) = 1$. 7

7. Find the recurrence relation that defines a_{n+2} in terms of preceding coefficients of a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ to the equation 7

$$(1 + x^2)y'' - 4xy' + 6y = 0.$$

8. Use the Laplace transform to solve the equation $y'' + 2y' + y = 4e^{-t}$, where $y(0) = 2$ and $y'(0) = -1$. | 7 | |

9. Find the general solution to system $\mathbf{y}' = \mathbf{A}\mathbf{y}$, where $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$. | 7 | |

TABLE OF LAPLACE TRANSFORM FORMULAS

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{1}{(n-1)!}\, t^{n-1}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \qquad \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$$

$$\mathcal{L}\left[\cos at\right] = \frac{s}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

Differentiation and integration

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = s\mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}\left[\frac{d^2t}{dt^2}f(t)\right] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$$

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

In the following formulas $F(s) = \mathcal{L}[f(t)]$, so $f(t) = \mathcal{L}^{-1}[F(s)]$.

$$\mathcal{L}\left[\int_0^t f(u)\ du\right] = \frac{1}{s}\mathcal{L}[f(t)] \qquad \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(u)\ du$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n\frac{d^n}{ds^n}\mathcal{L}[f(t)] \qquad \mathcal{L}^{-1}\left[\frac{d^nF(s)}{ds^n}\right] = (-1)^n t^n f(t)$$

Shift formulas

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a) \qquad \mathcal{L}^{-1}[F(s)] = e^{at}\mathcal{L}^{-1}[F(s+a)]$$

$$\mathcal{L}\left[u_a(t)f(t)\right] = e^{-as}\mathcal{L}[f(t+a)] \qquad \mathcal{L}^{-1}\left[e^{-as}F(s)\right] = u_a(t)f(t-a)$$

Here $u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$