AMAT 307 Fall 2006

DEs for Engineers

AMAT 307 Module 1 : Intro and 1st Order DEs

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Outline.

⇒ **Algebra: Exponential, Trigonometry, Complex Numbers, Polynomials**

⇒ **IntroDE: Terminology, Mathematical Models, Direction Fields**

⇒ **1st Order ODEs**

1 Algebra

1.1 Exponential

$$\begin{aligned} x^a x^b &= x^{a+b}, \quad x^a y^a = (xy)^a, \\ (x^a)^b &= x^{ab}, \quad x^{-a} = \tfrac{1}{x^a}, \\ e^{b\ln a} &= a^b, \quad \ln(xy) = \ln x + \ln y, \\ \ln(x^a) &= a\ln x, \quad \log_b(x) = \ln x/\ln b. \end{aligned}$$

1.2 Trigonometry

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B, \\ \cos(A+B) &= \cos A\cos B - \sin A\sin B, \\ \tan(A+B) &= \tfrac{\tan A+\tan B}{1-\tan A\tan B}, \\ \sin(A+B) &+ \sin(A-B) = 2\sin A\cos B, \\ \sin(A+B) &- \sin(A-B) = 2\cos A\sin B, \\ \cos(A+B) &- \cos(A-B) = -2\sin A\sin B, \\ \cos(A+B) &+ \cos(A-B) = 2\cos A\cos B, \end{aligned}$$

$$\begin{aligned} \sin(2x) &= 2\sin x\cos x, \\ \cos(2x) &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x, \\ \cos^2 x &= \tfrac{1}{2}(1+\cos 2x), \quad \sin^2 x = \tfrac{1}{2}(1-\cos 2x). \end{aligned}$$

1.3 Complex Numbers

$$\begin{aligned} i^2 &= -1, \quad i^3 = -i, \quad i^4 = 1, \\ |a+ib| &= \sqrt{a^2+b^2}. \end{aligned}$$

If $z = a + ib$, then $\bar{z} = a - ib$,

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a - ib}{a^2 + b^2}, \quad z \neq 0.$$

Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

1.4 Polynomials

Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

If $x = r$ is a root, i.e., $p(r) = 0$, then $(x - r)$ is a factor of $p(x)$, and conversely. If $a_n = 1$ and all a_i are integers, then any integer root r must divide a_0. Method of synthetic division for $p(x) \div (x - r)$. When all a_n are real numbers, then any complex roots occur in conjugate pairs $a \pm bi$.

2 IntroDEs

2.1 Terminology

Basic Terms: *ODE (Ordinary Differential Equations), PDE (Partial Differential Equations), Initial Condition (IC), Initial Value Problem (IVP), General Solution, Order of DE, Arbitrary Constants, Unique (specific) Solution, Linear ODE.*

Example:

$$a(t)y'' + b(t)y' + c(t)y = f(t)$$

is $2nd$ order linear DE. Dependent variable y, independent t.

General Principle: The order of the DE=number of arbitrary constants in the general solution (or the order of highest derivative in the DE).

2.2 Mathematical Models

Physical problem ⇒mathematical model⇒ solution.

Example 1:

$$s'' = -g$$

models motion of a falling body under constant gravitational field, g=gravity acceleration, $s = s(t)$-height above ground, t=time.

Example 2:

$$y' = ay + b$$

can model motion of a falling body with air resistance ($y = v$=velocity) or population growth with a predator (y=population), or Newton's cooling ($y = T$=temperature), or mixing problems (y=amount of solvent).

2.3 Direction Fields

For the first order DE

$$y' = f(t, x),$$

plot the slope of the tangent y' (a dash) at each point (t, y). When plotted for many points, the slopes indicate the solution curves-gives a visual picture of the specific solution passing through IC (t_0, y_0). An equilibrium solution is one that does not depend on t.

3 1st Order ODEs

3.1 General Form and Solutions

$$y' = f(t, y)$$

Other equivalent form, e.g.,

$$Mdt + Ndy = 0.$$

General Solution: must be one arbitrary constant;
Specific Soution: use IC to determine the value of the arbitrary constant.

3.2 Various Types and Methods of Solution

Linear:

$$y' + p(t)y = g(t),$$

solve by
a)

$$\mu(t) = e^{\int p(t)dt},$$

b)

$$y = \mu^{-1}(t)[\int \mu(t)g(t)dt + C].$$

Separable:

$$y' = g(t)p(y),$$

convert to

$$\frac{dy}{p(y)} = g(t)dt,$$

then $\int$ both sides.

Exact:

$$Mdt + Ndy = 0,$$

where

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

Let

$$F(t, y) = \int Mdt,$$

this yields an arbitrary function $g(y)$, which is determined using

$$\frac{\partial F}{\partial y} = N.$$

Solution is $F(t, y) = A$. If it's easier, one could also use
$F(t, y) = \int Ndy$, followed by $\frac{\partial F}{\partial t} = M$.

Homogeneous Coefficients: Can be written

$$\frac{dy}{dt} = f(\frac{y}{t}).$$

Substitute

$$y = tv \quad (v = \frac{y}{t}),$$

$$y' = v + tv',$$

and this converts to separable.

Bernoulli:

$$y' + p(t)y = f(t)y^n \quad (n \neq 0, 1).$$

Substitute

$$v = y^{1-n},$$

so

$$v' = (1 - n)y^{-n}y'.$$

Multiply DE by $(1 - n)y^{-n}$, converts to linear.

Integrating Factor:

Seek $\mu(t, y)$ so that

$$\mu Mdt + \mu Ndy = 0$$

is exact. No general rule to find μ, but often, if

$$(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t})/N = p(t) \quad (depends \quad only \quad on \quad t),$$

then

$$\mu(t) = e^{\int p(t)dt}$$

works. Similarly if

$$(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y})/M = h(y) \quad (depends \quad only \quad on \quad y).$$

Basic Theory

Picard's Theorem: If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous in an open rectangle $t_1 < t < t_2, \quad y_1 < y < y_2$, $(t_1, y_1 \quad \text{can} = -\infty, \quad \text{or} \quad t_2, y_2 \quad \text{can} = +\infty)$, and if (t_0, y_0) lies in this rectangle, then the initial value problem

$$y' = f(t,y), \quad y(t_0) = y_0,$$

has a unique solution lying in this rectangle.