AMAT 307 Fall 2006

DEs for Engineers

Review Module 3 : Linear Systems

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Outline

⇒ **Preliminaries**

⇒ **Complementary (Homogeneous) Solution**

⇒ **Particular Solution**

⇒ **Applications**

1 Preliminaries

Linear systems of first order DE's:

$$D\vec{y} = A\vec{y} + \vec{g}(t).$$

Here $\vec{y}$, $\vec{g}(t)$ are $n \times 1$ column vectors, A is an $n \times n$ matrix.

Homogeneous case is when $\vec{g}(t) = 0$, one then writes $\vec{y}_c$ for the solution, called the complementary (or homogeneous) solution.

As usual $\vec{y} = \vec{y}_c + \vec{y}_p$ (general solution = complementary solution+particular solution).

2 Complementary (Homogeneous) Solution

$$D\vec{y} = A\vec{y}.$$

First **find eigenvalues+eigenvectors for** A.

Eigenvalues: solve polynomial equation $p_A(\lambda) = 0$, where $p_A(\lambda) = det(A - \lambda I)$. There are n roots (counting multiplicities). To expand $det(A - \lambda I)$ the B-method, row & column operations are useful. Once an eigenvalue λ found, one solves the linear homogeneous equations $(A - \lambda I)\vec{v} = 0$ (or $B\vec{v} = 0$, where $B = A - \lambda I$) for the corresponding eigenvector(s), by reducing to row echelon form (REF). Note only row operations allowed here. At least one row of zeros must appear in the REF, and the number of 0 rows=the number of independent eigenvectors for λ.

Solution: For each eigenvalue λ and corresponding eigenvector $\vec{x}$, $\vec{y} = \vec{x}e^{\lambda t}$ is a solution of $D\vec{y} = A\vec{y}$.

The general solution is

$$\vec{y}_c = c_1\vec{x}_1 e^{\lambda_1 t} + c_2\vec{x}_2 e^{\lambda_2 t} + ... + c_n\vec{x}_n e^{\lambda_n t}.$$

For **complex eigenvalues** $\lambda = a \pm ib$ (they occur in conjugate pairs as long as A has real entries), take one of them, say $\lambda = a + ib$, & find corresponding eigenvector $\vec{x}$. Using $e^{(a+ib)t} = e^{at}(\cos t + i \sin t)$, and writing $\vec{x} = \vec{x}_1 + i\vec{x}_2$, we have $e^{(a+ib)t} = e^{at}[(\cos t\,\vec{x}_1 - \sin t\,\vec{x}_2) + i(\cos t\,\vec{x}_2 + \sin t\,\vec{x}_1)]$, and

$$\vec{y} = e^{at}[c_1(\cos t\,\vec{x}_1 - \sin t\,\vec{x}_2) + c_2(\cos t\,\vec{x}_2 + \sin t\,\vec{x}_1)]$$

gives two independent solutions.

For a **repeated eigenvalue** λ, say of multiplicity m, it is called "cheating" if fewer than m independent eigenvectors are obtained for λ. In this case, with $B = A - \lambda I$, solve $B^m\vec{x} = 0$ for the "generalized eigenvectors," it turns out there are always m independent generalized eigenvectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_m$. For each one, say $\vec{x}_i$,

$$\vec{y}_i = e^{\lambda t}(\vec{x}_i + B\vec{x}_i + \frac{t^2}{2!}B^2\vec{x}_i + ... + \frac{t^{m-1}}{(m-1)!}B^{m-1}\vec{x}_i)$$

is a solution and m independent solutions are thus obtained. Note short form $\vec{y}_i = e^{\lambda t}e^{tB}\vec{x}_i$.

3 Particular Solution

For $D\vec{y} = A\vec{y} + \vec{g}(t)$, use **variation of parameters**.

First find $\vec{y}_c$.

1) Write $\vec{y}_c = c_1\vec{x}_1 + ... + c_n\vec{x}_n, \quad c_i \in \mathbf{R}$ ($[\vec{x}_1 ... \vec{x}_n] = \Phi$) is called the fundamental solution matrix).

2) We seek $\vec{y}_p$ in the form $\vec{x}_p = u_1(t)\vec{x}_1 + ... + u_n(t)\vec{x}_n$

3) Find $u_1', ..., u_n'$ via equations $\vec{x}_1 u_1' + ... + \vec{x}_n u_n' = \vec{g}(t)$

4) Integrate to find $u_1(t), ..., u_n(t)$, then substitute in **2)**

In condensed matrix notation

$$\vec{y}_p = \Phi \int \Phi^{-1}\vec{g}(t)dt$$

(one can use same formula for $\vec{y}$ if the arbitrary constant $\vec{c} = \begin{bmatrix} c_1 \\ . \\ . \\ . \\ c_n \end{bmatrix}$ is included in the integral.

4 Applications

Physical systems with one or more dependent variables: mixing problems with two or more things being mixed in, coupled springs, LRC circuits with several loops, etc. Also higher order linear DE's convert to linear system via companion matrix.