

AMAT 307 Fall 2006
DEs for Engineers
Review Module: Laplace Transform
Courtesy: Dr. P. Zvengrowski
Outline

⇒ **Preliminaries**

⇒ **Laplace Transform**

⇒ **Initial Value Problems**

The Laplace Transform (LT) gives an alternative approach to solving IVPs of the type

$$L(D)(x) = g(t),$$

where $L(D)$ a linear operator with constant coefficients. It proceeds directly to the specific solution without first going to the general solution. It applies to continuous or even piecewise continuous $g(t)$ and is advantageous for such functions, e.g. pulse, sawtooth, etc.

1 Preliminaries

Partial fractions - know a few useful shortcuts, for example, in

$$\frac{2s + 1}{(s - 2)(s - 3)} = \frac{A}{s - 2} + \frac{B}{s - 3},$$

one has

$$A = \frac{2 \times 2 - 1}{2 - 3} = -5, \quad B = \frac{2 \times 3 + 1}{3 - 2} = 7.$$

Know the **general form**, for example,

$$\frac{s^3 + s + 1}{(s - 3)(s - 5)^2(s^2 + s + 2)} = \frac{A}{s - 3} + \frac{B}{s - 5} + \frac{C}{(s - 5)^2} + \frac{D}{(s - 5)^3} + \frac{Es + F}{s^2 + s + 2}.$$

Functions - understand meaning of function and how to shift. For example, if $f(t) = 3t + 1$, then $f(t + 2) = 3(t + 2) + 1 = 3t + 7$. If $f(t) = e^{-t}$, then $f(t + 2) = e^{-(t+2)} = e^{-2}e^{-t}$. If $f(t) = 5$, then $f(t + 2) = 5$.

Step functions-

$$h_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a \end{cases}$$

is the basic Heaviside step function. Also very useful are

$$1 - h_a(t) = \begin{cases} 1, & t < a, \\ 0, & t \geq a, \end{cases}$$

and for $a < b$ the “single unit pulse” function

$$h_a(t) - h_b(t) = \begin{cases} 1, & a \leq t < b, \\ 0, & \text{otherwise.} \end{cases}$$

These can be used to write any piecewise defined function.

2 Laplace Transform

Know the definition

$$\mathcal{L}[f(t)](s) := F(s) := \int_0^{+\infty} e^{-st} f(t) dt, \quad \mathcal{L}^{-1}[F(s)](t) = f(t).$$

Basic property is linearity, and be familiar with some basic transforms:

$f(t)$	1	$e^{\lambda t}$	$\cos bt$	$\sin bt$	t^n	$h_a(t)$
$F(s)$	$1/s$	$1/(s - \lambda)$	$s/(s^2 + b^2)$	$b/(s^2 + b^2)$	$n!/s^{n+1}$	e^{-as}/s

There are also convergence conditions. Tables of \mathcal{L} usually given.

3 Initial Value Problems

First take \mathcal{L} of both sides, using 1'st differentiation formula (below) and rules for \mathcal{L} to get $\mathcal{L}[g(t)]$. Then solve for $\mathcal{L}[y]$, simplify with partial fractions where needed. Finally, take \mathcal{L}^{-1} to get y . Useful rules in these procedures (also given usually with tables):

$$1'st \text{ diff} : \mathcal{L}[D^n y] = s^n \mathcal{L}[y] - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - \dots$$

$$1'st \text{ shift} : \mathcal{L}[e^{at} f(t)] = F(s - a), \quad \mathcal{L}^{-1}F(s - a) = e^{at} f(t)$$

$$2'nd \text{ diff} : \mathcal{L}[t^n f(t)] = (-1)^n D^n F(s), \quad \mathcal{L}^{-1}D^n F(s) = (-1)^n \mathcal{L}[t^n f(t)]$$

$$2'nd \text{ shift} : \mathcal{L}[h_a(t)f(t)] = e^{-as} \mathcal{L}[f(t + a)], \quad \mathcal{L}^{-1}[e^{-as}F(s)] = h_a(t)f(t - a)$$

$$\text{convolution} : (f * g)(t) = \int_0^t f(t - u)g(u)du - \text{linear in } f \text{ and in } g,$$

$$\text{and } f * g = g * f, \quad (f * g) * h = f * (g * h).$$

Convolution is useful because

$$\mathcal{L}[(f * g)(t)] = F(s)G(s), \quad \text{or equivalently} \quad \mathcal{L}^{-1}(FG) = f * g.$$

For IVP at $t = a$, change of variable $\tau = t - a$, reduces it to $\tau = 0$.