University of Calgary
Department of Mathematics and Statistics
Amat 307 Midterm Test
Time: 90 minutes
NO CALCULATORS, NO AIDS

ENTER ANSWERS IN THE BOXES

Surname	Other Names	Lec. #

Find a 1-parameter family of solutions to the following four first order differential equations.

[2] 1. $y' - ty = 0$

Integrating factor (IF) is $e^{-t^2/2}$

so $y(t) = a e^{t^2/2}$, $a \in \mathbb{R}$.

[2] 2. $y' + 3t^2 y = 3t^2 e^{-t^3}$

$IF = e^{t^3}$ so $(e^{t^3} y)' = 3t^2$

$e^{t^3} y = t^3 + a$

$\boxed{y(t) = t^3 e^{-t^3} + a e^{-t^3}, \; a \in \mathbb{R}}$

[2] 3. $[\sin(y)2t]dy - \cos(y)dt = 0,$

sep.

$2\tan(y)\,dy = \frac{1}{t}\,dt$

$2\ln|\sec(y)| = \ln|t| + c$

$\boxed{\sec^2(y) = at, \; a \in \mathbb{R}.}$

[3] 4. $y' - t^{-1}y = -y^2 t$

Bernoulli

$\boxed{y = \frac{3t}{t^3 + a} \; ; \; a \in \mathbb{R}}$

$-y^{-2}y' + t^{-1}y^{-1} = t$. Let $u = y^{-1}$

so $\frac{du}{dt} = -y^{-2}y'$

$u' + t^{-1}u = t$ linear with $IF = e^{\ln(t)} = t$.

$(tu)' = t^2$; $tu = \frac{t^3}{3} + c$; $u = \frac{t^2}{3} + ct^{-1}$; $\frac{1}{y} = \frac{t^3 + a}{3t}$

[3] 5. Solve the initial value problem $(e^{2y} + t^2 y)y' + ty^2 + \cos(t) = 0$, $y(\pi/2) = 0$.

$\frac{\partial(e^{2y} + t^2 y)}{\partial t} = 2ty = \frac{\partial(ty^2 + \cos(t))}{\partial y}$ so exact. $\boxed{\frac{e^{2y}}{2} + \frac{t^2y^2}{2} + \sin(t) = \frac{3}{2}.}$

$c = F(t,y) = \frac{1}{2}e^{2y} + \frac{t^2y^2}{2} + \sin(t)$

When $t = \pi/2$ and $y = 0$ get $c = \frac{1}{2} + 1 = 3/2$

In each of the following three differential equations find the roots of the charactistic polynomial and the general solution of the differential equation in terms of real valued functions. all 2^{nd} order linear, constant coefficients.

[2] 6. $y'' - 5y' + 6y = 0$

Characteristic polynomial (= CP) is
$\lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$; has roots $\{2, 3\}$ $\boxed{ae^{2t} + be^{3t};\ a, b \in \mathbb{R}.}$

[3] 7. $y'' - 2y' + 2y = 0$

CP is $\lambda^2 - 2\lambda + 2$ which has roots
$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ $\boxed{ae^t\cos(t) + be^t\sin(t);\ a, b \in \mathbb{R}.}$

[2] 8. $y''' - 5y'' + 8y' - 4y = 5e^t$

CP is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda-1)(\lambda^2 - 4\lambda + 4)$
$= (\lambda-1)(\lambda-2)^2$ has roots $\{1, 2, 2\}$. $\boxed{ae^t + be^{2t} + cte^{2t} + 5te^t;\ a, b, c \in \mathbb{R}.}$

Need to look for a particular sol'n y_p of the form ate^t.

$y_p = ate^t$
$y_p' = ae^t + ate^t$
$y_p'' = 2ae^t + ate^t$
$y_p''' = 3ae^t + ate^t$

$-4y_p = -4ate^t$
$8y_p' = 8ae^t + 8ate^t$
$-5y_p'' = -10ae^t - 5ate^t$
$y_p''' = 3ae^t + ate^t$
$5e^t = ae^t$ so $a = 5$.

[4] 9. The equation $y''+4y'-5y = 2e^{-t}-8te^{-t}+5t$ has a particular soution $y_p = te^{-t}-t-4$.

(a) What is the general solution?

CP is $\lambda^2+4\lambda-5 = (\lambda+5)(\lambda-1)$; has roots $\{-5, 1\}$. so gen. sol'n to associated homog. eq is $ae^{-5t}+be^t$.

$\boxed{ae^{-5t}+be^t+te^{-t}-t-4 \quad a,b\in\mathbb{R}.}$

(b) What is the unique solution satisfying the initial conditions $y(0) = 1$, $y'(0) = -1$?

$$\begin{cases} 1 = y(0) = a+b-4 \\ -1 = y'(0) = -5a+b+1-1 \end{cases}$$

$\boxed{e^{-5t}+4e^t+te^{-t}-t-4}$

$$\begin{cases} 5 = a+b \\ -1 = -5a+b \end{cases}$$ so $a=1$, $b=4$.

[4] 10. Given the equation $y''+y' = (1+e^t)^{-1}$.

(a) Find a fundamental set of solutions of the associated homogeneous equation and compute its Wronskian. $\lambda^2+\lambda$ = C.P. has roots $\{0,-1\}$.

$\{h_1 = 1,\ h_2 = e^{-t}\}$ is a fund set of sol's. $W(h_1,h_2) = \begin{vmatrix} 1 & e^{-t} \\ 0 & -e^{-t} \end{vmatrix} =$

$\boxed{-e^{-t}}$

$= -e^{-t}$

(b) Find a particular solution.

$y_p = c_1(t)h_1(t) + c_2(t)h_2(t)$.

$\boxed{t-\ln(1+e^t) - [\ln(1+e^t)]e^{-t}}$

$$c_1' = \frac{\begin{vmatrix} 0 & e^{-t} \\ (1+e^t)^{-1} & -e^{-t} \end{vmatrix}}{W} = (1+e^t)^{-1}$$

$$c_2' = \frac{\begin{vmatrix} 1 & e^{-t} \\ 0 & (1+e^t)^{-1} \end{vmatrix}}{W} = -\frac{(e^t)}{(1+e^t)} \quad \text{so} \quad c_2 = -\ln(1+e^t)$$

$$c_1 = \int \frac{1}{1+e^t} = \int \frac{1+e^t}{1+e^t} - \frac{e^t}{1+e^t} = t - \ln(1+e^t)$$ (also substitution and partial fractions will work)

[4] 11. The third order nonhomogeneous equation $y''' - e^t y'' + ty' - 6y = te^t$ can be viewed as a nonhomogeous system of order 3: $D\overrightarrow{y} = A\overrightarrow{y} + \overrightarrow{q}$.

(a) What is the matrix A equal to?

$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ with $y_1 = y$, $y_2 = y'$, $y_3 = y''$ so $D\vec{y} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} =$

$\begin{bmatrix} y_2 \\ y_3 \\ 6y_1 - ty_2 + e^t y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ te^t \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -t & e^t \end{bmatrix}}$$

(b) What is $\overrightarrow{q}$ equal to?

$\vec{q} = \begin{bmatrix} 0 \\ 0 \\ te^t \end{bmatrix}$

[4] 12. A tank continuously mixes 200 litres of sugar water which initially has a concentration of 70 grams/litre. To increase this concentration a sugar solution with a concentration of 120 grams/litre is pumped in at a rate of 1 litre/minute and the mixture is then pumped out at a rate of 1 litre/minute. When does the tank contain water with a sugar concentration of 100 grams/litre?

$y(t) =$ amount of sugar in grams in tank at time t in minutes.

$\boxed{200 \ln(5/2) \text{ min.}}$

$y' = \left[120\ g/\ell\right]\left[1\ \ell/\text{min}\right] - \left[\frac{y(t)}{200}\right] g/\ell \left(1\ \ell/\text{min}\right)$

$y' = 120 - \frac{y}{200}$; $y' + \frac{y}{200} = 120$

I.F. $= e^{t/200}$ so $y = a e^{-t/200} + (120)(200)$

$y(0) = (70\ g/\ell)(200\ \ell) = 70(200) = a + (120)(200)$

so $a = (70-120)(200) = (-50)(200)$.

Find t with $\frac{y(t)}{200} =$ concentration at time t $= 100$

Now $\frac{y(t)}{200} = \left[(-50)e^{-t/200} + 120\right]$ so find t with

$-50 e^{-t/200} + 120 = 100$; $e^{-t/200} = \frac{-20}{-50} = \frac{2}{5}$; $\frac{t}{200} = \ln(5/2)$

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