AMAT 307
DEs for Engineers
AMAT 307 Module 2 : Linear Differential Equations
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Outline

1 Algebra:

Useful to recall determinants and methods to find roots (zeros) of polynomials, e.g. factorization of polynomials.

• If a is a root of a polynomial $p(\lambda)$, then $\lambda - a$ is a factor. The remaining factor can be found by long division, or (more quickly) synthetic division.

• For any polynomial $p(\lambda) = a_n\lambda^n + ... + a_1\lambda + a_0$, assuming a_j are real, complex roots occur in conjugate pairs $x \pm iy$.

• For $p(\lambda)$ as above, assuming all a_j are integers, then any rational root s/t (s, t integers) satisfies: s divides a_0, t divides a_n.

2 Terminology:

Linear DE of order n has form $a_n(t)D^n y + ... + a_1(t)Dy + a_0(t)y = g(t)$, for short write $L(D) = a_n(t)D^n + ... + a_1(t)D + a_0(t)$, then $L(D)y = g(t)$ is shorthand (operator) notation for the DE. If $g(t) = 0$, the DE is called homogeneous. If each function $a_j(t) = a_j$, a constant, we say the DE has constant coefficients and write $L(D) = p(D) = a_n D^n + ... + a_1 D + a_0$, a polynomial in D. The polynomial $p(\lambda) = a_n\lambda^n + ... + a_1\lambda + a_0$ is then called the characteristic polynomial of the DE.

3 Superposition Principle:

(1) If y_1, y_2 are solutions of the homogeneous DE $L(D)y = 0$, then so is $y = Ay_1 + By_2$ a solution.

(2) If y_c is any solution of $L(D)y = 0$ and y_p any solution of $L(D) = g(t)$, then $y = y_c + y_p$ is a solution of $L(D)y = g(t)$.

(3) Strategy for finding the general solution of $L(D)y = g(t)$ is to find the general solution y_c for $L(D)y = 0$ (with n arbitrary constants), then find any particular solution y_p for $L(D)y_p = g(t)$, and finally take $y = y_c + y_p$ as the general solution for $L(D)y = g(t)$.

4 Linear Independence of Functions:

The Wronskian $W(f_1, ..., f_n) = W(t)$ is the $n \times n$ determinant having first row $f_1(t), ..., f_n(t)$, second row $f_1'(t), ..., f_n'(t)$, etc. If $W(t) \neq 0$ (for any value of t) then $f_1, ..., f_n$ are linearly independent. For functions $f_1, ..., f_n$ arising as solutions of an $n'th$ order linear homogeneous DE, they are linearly independent iff $W(t) \neq 0$. In that case $\{f_1, ..., f_n\}$ is called a fundamental set of solutions.

Abel's Theorem can often simplify the calculation of Wronskians. If $\{f_1, ..., f_n\}$ is a fundamental solution set of the DE $1 \cdot D^n y + p_{n-1}D^{n-1}y + \cdots + p_1(t)Dy + p_0(t)y = 0$, each $p_j(t)$ continuous on an interval $a < t < b$, then their Wronskian is given by

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p_{n-1}(s)}ds \ .$$

5 Constant Coefficients:

For $L(D) = p(D) = a_n D^n + ... + a_1 D + a_0$, to solve $p(D)y = 0$, find the roots of the characteristic equation $p(\lambda) = 0$. For each root λ_i, $e^{\lambda_i t}$ is a solution. By superposition $y = \sum c_i e^{\lambda_i t}$ is a solution. If all roots are distinct (n roots) this is the general solution. If some root is repeated, e.g., $\lambda_1, \lambda_1, \lambda_1$, then $c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_3 t^2 e^{\lambda_1 t}$ is a solution $\Rightarrow$ general solution. For a complex conjugate pair $a \pm bi$, $c_1 e^{at} \cos bt + c_2 e^{at} \sin bt$ is a solution.

6 Exponential Shift Formula:

Useful theoretical tool

$$L(D)(e^{\lambda t}y) = e^{\lambda t}L(D+\lambda)y.$$

7 Applications:

Vibrating spring with friction $\quad my'' + \gamma y' + ky = F_a(t)$

LRC electrical circuit $\quad LQ'' + RQ' + \dfrac{1}{C}Q = V_s(t)$

Mathematically these applications are equivalent to the DE $\;y'' + ay' + by = g(t)$, where $a \geq 0$, $b > 0$. The general behaviour of the solutions (for the homogeneous case $g(t) = 0$) depends on the discriminant $\Delta = a^2 - 4b$. If $\Delta > 0$ we have the *overdamped* case = exponential decay , if $\Delta = 0$ the *critically damped* case = temporary growth followed by exponential decay, and $\Delta < 0$ the *underdamped* case = oscillations.

8 Particular Solution, Undetermined Coefficients

Applicable only for constant coefficients linear DE $p(D)y = g(t)$, when $g(t)$ is formed from sums of products of functions t^n $(n \geq 0)$, e^{at}, $\sin bt$, $\cos bt$.

1) find y_c, i.e., solve $P(D)y = 0$, n arbitrary constants,

2) set up test y_p based on the terms in $g(t)$, $g'(t)$, $g''(t)$,...,

3) if any terms in test y_p are part of y_p, multiply the affected terms (and only these terms) by the least t^m that eliminates this overlap,

4) substitute y_p into the DE, compare like terms to find the unknown constants in y_p.

9 Particular Solution, Variation of Parameters:

A very general method

$L(D)y = g(t)$, where $L(D)$ is any linear differential operator and $g(t)$ any function

1) Make sure DE in standard form $L(D) = 1 \times D^n + ... + a_1(t)D + a_0(t)$, solve for y_c,

2) Write (taking $n = 3$ as example) $y_c = c_1h_1(t) + c_2h_2(t) + c_3h_3(t)$,

3) let $y_p = u_1(t)h_1(t) + u_2(t)h_2(t) + u_3(t)h_3(t)$,

4) solve for u_1', u_2', u_3' from the equations

$$\begin{cases} h_1 u_1' + h_2 u_2' + h_3 u_3' &= 0 \\ h_1' u_1' + h_2' u_2' + h_3' u_3' &= 0 \\ h_1'' u_1' + h_2'' u_2' + h_3'' u_3' &= g(t), \end{cases}$$

5) integrate to get u_1, u_2, u_3, then substitute back into 3).

10 Cauchy-Euler Type:

$L(D) = t^n \times D^n + a_{n-1}t^{n-1}D^{n-1} + ... + a_1 tD + a_0$. Can solve $L(D) = 0$ using $y = t^r$ as test solution. For $n = 2$, say $L(D) = t^2D^2 + atD + b$, the (quadratic) equation for r is $r^2 + (a-1)r + b = 0$. If repeated roots r_1, r_1, get $y_c = At^{r_1} + Bt^{r_1} \ln t$. If complex roots $r = c \pm di$, get $y_c = At^c \cos(d \ln t) + Bt^c \sin(d \ln t)$. For y_p use variation of parameters, and do not forget to first divide the DE by t^n.