AMAT 307

DEs for Engineers

Review Module: Infinite Series Solutions of DEs

Courtesy: Dr. P. Zvengrowski

Outline

⇒ **Preliminaries**

⇒ **Solution at Ordinary Points**

⇒ **Solution at Regular Singular Points**

⇒ **Initial Conditions**

Infinite series solutions can be applied to a wide class of linear DE's

$$L(D) = g(t).$$

For simplicity we take $L(D)$ $2'$nd order and the homogeneous case $g(t) = 0$, the general case is quite similar ($g(t)$ would first also be expanded into a power series). Standard form:

$$y'' + p(t)y' + q(t)y = 0. \qquad (*)$$

1 Preliminaries

(A) **Infinite sequences.** An infinite sequence is written $\{a_n\}$, be familiar with terms such as increasing, decreasing, positive, negative, bounded above, bounded below, bounded, alternating, and $\lim_{n\to\infty} a_n$, if it exists. When this limit exists one calls $\{a_n\}$ convergent, otherwise divergent.

(B) **Infinite series.** An infinite series is written $\sum_{j=0}^{\infty} a_j$. The n'th partial sum is defined to be $s_n = \sum_{j=0}^{n} a_j$, and one calls the infinite series convergent iff $\{s_n\}$ is a convergent sequence, otherwise divergent. In case it is convergent one defines $\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} s_n$, in other words the sum of an infinite series is the limit of its partial sums. One calls the infinite series absolutely convergent if $\sum_{j=0}^{\infty} |a_j|$ converges, absolute convergence always implies convergence. An infinite series that is convergent but not absolutely convergent, such as $\sum_{j=1}^{\infty} (-1)^j/j$, is called conditionally convergent.

The most important convergence tests are the ratio test, comparision test, root test, integral test, and (for alternating series) the alternating series test. In order to use the comparision test one needs to know a few basic series such as the geometric series $\sum_{j=0}^{\infty} r^j$, convergent iff $|r| < 1$, and the p-series $\sum_{j=1}^{\infty} 1/j^p$, convergent iff $p > 1$.

(C) **Power series.** A power series centred at t_0 has the form

$$\sum_{n=0}^{+\infty} a_n (t - t_0)^n,$$

radius of convergence called R. Know some standard power series like

$$e^t, \quad \sin t, \quad \cos t, \quad \frac{1}{1-t}, \quad \ln(1+t), \quad \arctan t.$$

It is useful (and no more difficult) to think of t, t_0, y as complex numbers henceforth. The power series is absolutely convergent in the disc $|t - t_0| < R$.

Definition. We say $f(t)$ is **analytic** at t_0, if

$$f(t) = \sum_{n=0}^{+\infty} a_n (t - t_0)^n, \quad |t - t_0| < R, \quad R > 0.$$

Often one takes $t_0 = 0$.

Fact:

$$\sum_{n=0}^{+\infty} a_n t^n = 0, \text{for all} \quad t, \quad \text{iff each} \quad a_n = 0, \quad n \geq 0.$$

Summations-be able to use 'tricks' such as shifting the indicies:

$$\sum_{n=3}^{+\infty} c_n = \sum_{n=0}^{+\infty} c_{n+3} = \sum_{i=0}^{+\infty} c_{i+3}.$$

2 Solution at Ordinary Points

Definition: t_0 is an **ordinary point** of (*) iff $p(t), \quad q(t)$ are analytic at $t = t_0$.

Main Theorem: *At an ordinary point t_0, (*) has two linearly independent analytic solutions of the form*

$$y = \sum_{n=0}^{+\infty} a_n (t - t_0)^n,$$

with $R \geq$ the distance of t_0 to the nearest singular point of p and q, in the complex plane $\mathbb{C}$.

Method of Solution: *Substitute (1) into the DE (*), using shifting trick as needed to put all terms into form $\sum_{n=0}^{+\infty}(...)t^n$. Now combine and set the coefficients of each t^n equal to 0.*

This leads to the **recurrence relation**

$$a_{n+2} = f(a_n, a_{n+1}).$$

Let $a_0 = A, \quad a_1 = B \in \mathbb{R}$, then recurrence relation determines $a_2, a_3, a_4, ...$ in terms of A, B and thus the general solution.

3 Solution at Regular Singular Points

Definition: We say $t = t_0$ is a **regular singular point** if both $(t - t_0)p(t)$, $(t - t_0)^2 q(t)$ are analytic at $t = t_0$.

In this case the Frobenius method leads to analytic solutions about t_0, very similar to the ordinary point method with one major difference, use

$$y = (t - t_0)^r \sum_{n=0}^{+\infty} a_n (t - t_0)^n. \tag{1'}$$

The exponent r can be determined

(a) from the lowest power $(t - t_0)^r$ in working out the series solution.

(b) from $r(r-1) + rp_0 + q_0 = 0$,

where

$$p_0 = \lim_{t \to t_0} (t - t_0) p(x), \quad q_0 = \lim_{t \to t_0} (t - t_0)^2 q(t).$$

At least one solution with R as before is obtained this way, and two are obtained when the two roots r_1, r_2 do not differ by an integer. If $r_1 - r_2 =$ *integer*, extra work will be needed to find 2nd independent solution.

Easiest example: Cauchy-Euler type $t^2 y'' + aty' + by = 0 \quad (t > 0)$, solution is $y = t^r$, i.e. $y = At^{r_1} + Bt^{r_2}$ is a general solution.

Other examples (important in mathematical physics, engineering): Bessel functions, Legendre polyomials, Laguerre polynomials, Hermite polynomials, Tschebyshev polynomials, hypergeometric functions.

4 Initial Conditions

For a general solution

$$y = A + Bt + a_2 t^2 + a_3 t^3 + ...,$$

and IC $y(0) = a$, $y'(0) = b$, one has $A = a$, $B = b$ for the specific (unique) solution.

Caution: For a 3′rd order DE with $y = A + Bt + Ct^2 + a_3 t^3 + ...$, with IC $y(0) = a$, $y'(0) = b$, $y''(0) = c$, this would be $A = a, \quad B = b, \quad C = c/2!$, with the obvious generalization to 4'th and higher order DEs.