AMAT 307 DEs for Engineers

Review Module: Laplace Transform

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Outline

⇒ **Preliminaries**

⇒ **Laplace Transform**

⇒ **Initial Value Problems**

The Laplace Transform (LT) gives an alternative approach to solving IVPs of the type

$$L(D)(y) = g(t),$$

where $L(D)$ a linear operator with constant coefficients. It proceeds directly to the specific solution without first going to the general solution. It applies to continuous or even piecewise continuous $g(t)$ and is advantageous for such functions, e.g. pulse, sawtooth, etc.

1 Preliminaries

Partial fractions - know a few useful shortcuts, for example, in

$$\frac{2s+1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3},$$

one has

$$A = \frac{2 \times 2 - 1}{2-3} = -5, \quad B = \frac{2 \times 3 + 1}{3-2} = 7.$$

Know the **general form**, for example,

$$\frac{s^3+s+1}{(s-3)(s-5)^2(s^2+s+2)} = \frac{A}{s-3} + \frac{B}{s-5} + \frac{C}{(s-5)^2} + \frac{D}{(s-5)^3} + \frac{Es+F}{s^2+s+2}.$$

Functions - understand meaning of function and how to shift. For example, if $f(t) = 3t+1$, then $f(t+2) = 3(t+2)+1 = 3t+7$. If $f(t) = e^{-t}$, then $f(t+2) = e^{-(t+2)} = e^{-2}e^{-t}$. If $f(t) = 5$, then $f(t+2) = 5$.

Step functions-

$$u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a \end{cases}$$

is the basic Heaviside step function. Also very useful are

$$1 - u_a(t) = \begin{cases} 1, & t < a, \\ 0, & t \geq a, \end{cases}$$

and for $a < b$ the "single unit pulse" function

$$u_a(t) - u_b(t) = \begin{cases} 1, & a \leq t < b, \\ 0, & \text{otherwise.} \end{cases}$$

These can be used to write any piecewise defined function.

2 Laplace Transform

Know the definition

$$\mathcal{L}[f(t)](s) := F(s) := \int_0^{+\infty} e^{-st} f(t)dt, \quad \mathcal{L}^{-1}[F(s)](t) = f(t).$$

Basic property is linearity, and be familiar with some basic transforms:

$f(t)$	1	$e^{\lambda t}$	$\cos bt$	$\sin bt$	t^n	$u_a(t)$
$F(s)$	$1/s$	$1/(s-\lambda)$	$s/(s^2+b^2)$	$b/(s^2+b^2)$	$n!/s^{n+1}$	e^{-as}/s

There are also convergence conditions. Tables of $\mathcal{L}$ usually given.

3 Initial Value Problems

First take $\mathcal{L}$ of both sides, using 1'st differentiation formula (below) and rules for $\mathcal{L}$ to get $\mathcal{L}[g(t)]$. Then solve for $\mathcal{L}[y]$, simplify with partial fractions where needed. Finally, take $\mathcal{L}^{-1}$ to get y. Useful rules in these procedures (also given usually with tables):

$1'\text{st} \quad \text{diff} : \mathcal{L}[D^n y] = s^n \mathcal{L}[y] - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - ...$

$1'\text{st} \quad \text{shift} : \mathcal{L}[e^{at}f(t)] = F(s-a), \quad \mathcal{L}^{-1}F(s-a) = e^{at}f(t)$

$2'\text{nd} \quad \text{diff} : \mathcal{L}[t^n f(t)] = (-1)^n D^n F(s), \quad \mathcal{L}^{-1}D^n F(s) = (-1)^n \mathcal{L}[t^n f(t)]$

$2'\text{nd} \quad \text{shift} : \mathcal{L}[u_a(t)f(t)] = e^{-as}\mathcal{L}[f(t+a)], \quad \mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)$

convolution : $(f * g)(t) = \int_0^t f(t-u)g(u)du$ – linear in f and in g,

and $f * g = g * f, \quad (f * g) * h = f * (g * h)$.

Convolution is useful because

$$\mathcal{L}[(f * g)(t)] = F(s)G(s) \ , \quad \text{or equivalently} \quad \mathcal{L}^{-1}(FG) = f * g \ .$$

For IVP at $t = a$, change of variable $\tau = t - a$, reduces it to $\tau = 0$.