Applied Mathematics 309 Assignment #1

due date: February 6, 2004

1. Given two linearly independent vectors a and b in $\mathbb{R}^3$, show that the frame $\{a, b, a \times b\}$ is a positively oriented frame.

2. Let a and b be linearly independent vectors in $\mathbb{R}^3$ with $a \cdot b = 0$. Find all vectors v such that $a \times v = b$.

3. Find the point(s) of maximum curvature on the graph of $y = \sinh x$.

4. Let a disc of radius a roll along the bottom of the line $y = 2a$ (so the centre of the disc moves along the line $y = a$.

   (a) Give a parametrization of the curve c traced out by the point of the disc that passes through the origin $(0, 0)$.

   (b) Give an arclength parametrization of the cycloid you found in the previous part, with the condition that $c(s = 0) = (0, 0)$.

   (c) Compute the curvature of the cycloid.

   (d) Suppose a particle of mass m slides along this curve without friction and under the influence of gravity (i.e. the potential energy at $(x(s), (y(s))$ is mgy, where g is the acceleration of gravity). The time τ it takes for the particle to slide along the curve from rest to the origin is given by the integral **

$$\tau = \int dt = \int \frac{dt}{ds}\, ds = \int \frac{1}{v}\, ds$$

   where v is the velocity. Using conservation of energy, show that the time τ it takes for the particle to slide is *independent of the place it starts.* That is to say, the period of oscillation is independent of the amplitude. This is why the cycloid is also known as a tautochrone.

5. (Extra for experts). Find a curve $c(s)$, $s \in (-\pi/2, \pi/2)$, if the curvature and torsion are given by

$$\kappa(s) = \cos s, \qquad \tau(s) = \sin s.$$