1. Suppose we define the eccentric anomaly by the equation $s = p \cdot q - 2ht$, and let $r = |q|$ be the radial distance. Show that

$$\frac{ds}{dt} = \frac{1}{r},$$

so that s really is the eccentric anomaly. Note: As old as the Kepler problem is, this trick was only discovered by Souriau in 1983!

2. Show that the eccentricity vector

$$e = \frac{1}{k}(p \times j) - \frac{1}{r}q$$

is a constant of motion of the Kepler problem.

3. The effective potential and one-dimensional motion. If we just study the radial part of the motion of a planet in the Kepler problem, it turns out that the radial distance r behaves as if it was moving in the potential field

$$V(r) = \frac{\mu^2}{2r^2} - \frac{k}{r}.$$

Sketch a graph of $V(r)$ for $r > 0$. Where does $V'(r) = 0$? To what does this motion correspond to? The constant μ is the length of the angular momentum vector, $\mu = |j|$.

4. Let $r = \sqrt{x^2 + y^2}$. For what value(s) of n is r^n a harmonic function? That is to say, $u(x, y) = r^n$ satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

5. Show that $u(x, t) = 2k^2 \cosh^{-2}(k(x - 4k^2 t))$ where $k \geq 0$ is a solution of the KdV equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

It is a good exercise to plot this curve on a computer so that you get an accurate idea of what it looks like, as this is the famous *soliton*.