Applied Mathematics 309
Final Winter 2004
Lecture Sections L01/L02

A single $8\frac{1}{2} \times 11$ formula sheet is allowed, but no other aids.

1. Answer either True or False (*not* T or F). Here $\mathbf{v}, \mathbf{w}, \mathbf{x}$ are vectors in $\mathbb{R}^3$. [10]

   (a) A curve $\mathbf{r}(t)$ with constant curvature and torsion (both positive) is a circular helix. __________

   (b) $\mathbf{v} \bullet (\mathbf{x} \times \mathbf{w}) = (\mathbf{x} \times \mathbf{v}) \bullet \mathbf{w}$. __________

   (c) $\mathbf{v} \bullet (\mathbf{x} \times \mathbf{v}) = \mathbf{0}$. __________

   (d) A point of maximal curvature on a curve is also a point of maximal normal acceleration a_N. __________

   (e) The hyperboloid of two sheets $-x^2/a^2 - y^2/b^2 + z^2/c^2 = 1$ is a ruled surface. __________

   (f) The set of all points $\mathbf{p}$ such that $\mathbf{p} \bullet \langle 1, 2, 2 \rangle = \|\mathbf{p}\|$ is a cone. __________

   (g) For any smooth vector &eld $\mathbf{F}$, $\text{curl}(\text{curl}(\mathbf{F})) = \mathbf{0}$. __________

   (h) For any smooth function $f(x, y, z)$, $\text{curl}(\text{grad}(f)) = \nabla \times (\nabla f) = \mathbf{0}$. __________

   (i) A smooth curve c with zero torsion lies in a plane. __________

   (j) If D is the disc $x^2 + y^2 \leq 4$, and $f(x, y)$ has a strict local minimum at the origin $(0, 0)$, then $f(0, 0)$ is the absolute minimum of f on D. __________

2. Let $\mathbf{r}(t) = \langle t, t^2/2, t^3/3 \rangle$ be a smooth curve in $\mathbb{R}^3$. Find the maximum torsion ofthe curve. [10]

3. Find the maximum and minimum values of the function $f(x, y) = 2xy$ [10] on the closed disc $x^2 + y^2 \leq 9$.

4. What is the largest volume of a closed rectangular box of surface area 16 cm^2, the sum of whose edge lengths is 20 cm? Note: the answer is *not* a cube. [10]

5. Let a ! at body of constant mass density k cover the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in the plane, and let $a \geq 0$. [10]

(a) Find the moment of inertia of the body about the line $x = -a$.

(b) Find a point (x_1, y_1) in the body such that the moment of inertia of the body about the y-axis is the product of the mass and the square of the distance from (x_1, y_1) to the axis.

6. Compute, by any method you prefer, the line integral [10]

$$\int_c (x - y^3)\,dx + (x^3 + y^3)\,dy$$

where c is the positively oriented boundary of the quarter disc

$$Q = \{(x, y) |\, 0 \leq x^2 + y^2 \leq a^2,\ x \geq 0,\ y \geq 0\}.$$

7. Sketch the solid lying in the &rst octant, inside the cylinder $x^2 + y^2 = 2x$, and under the plane $z = y$. Also, &nd the volume of the solid. [10]

8. Consider the vector &eld $\mathbf{G}(x, y, z) = \langle x, y - z, -2z \rangle$. [10]

(a) Compute the divergence of $\mathbf{G}$.

(b) Find another vector &eld $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ with $\mathbf{G} = \text{curl}(\mathbf{F})$. As a start, you could try setting $Q \equiv 0$ and $R = xy$.

9. Let G be a solid region in $\mathbb{R}^3$, with boundary $\partial G = S$, and let $\mathbf{r} = \langle x, y, z \rangle$. Show that [10]

$$\iint_S \mathbf{r} \bullet \mathbf{n} \, dS = 3V(G),$$

where $\mathbf{n}$ is the outward normal to S and $V(G)$ is the volume of G.

10. Let S be the surface that is the portion of the sphere of radius a, centred at the origin, that lies above the cone $z^2 = x^2 + y^2$. Assume that S is oriented by the upward normal $\mathbf{n}$. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle x^2 - y^3 z, x^3 z, xyz \rangle$. Compute [10]

$$\iint_S \operatorname{curl}(\mathbf{F}) \bullet \mathbf{n}\, dS$$