Applied Mathematics 309
Final Winter 2003
Lecture Sections L01/L02

A single $8\frac{1}{2} \times 11$ formula sheet is allowed, but no other aids.

1. Answer (a) to (e) either vector, scalar, or meaningless and (f) to (j) either True or False (do not abbreviate). Here $\mathbf{v}, \mathbf{w}, \mathbf{x}$ are vectors in $\mathbb{R}^3$. [10]

   (a) $(\mathbf{v} \bullet \mathbf{x}) \bullet \mathbf{w}$ __________

   (b) $(\mathbf{v} \bullet \mathbf{x}) \times \mathbf{w}$ __________

   (c) $(\mathbf{v} \times \mathbf{x}) \bullet \mathbf{w}$ __________

   (d) $(\mathbf{v} \times \mathbf{x}) \times \mathbf{w}$ __________

   (e) $\mathbf{v} \times \mathbf{x} \times \mathbf{w}$

   (f) On any smooth curve $\mathbf{c}(t)$ with velocity $\mathbf{v} \neq \mathbf{0}$, normal $\mathbf{N}$, acceleration $\mathbf{a}$, curvature κ and speed v, $\mathbf{N} \bullet \mathbf{a} = \kappa v^2$. __________

   (g) For any smooth function $f(x, y)$, $f_{xy} f_y = f_x f_{yy}$. __________

   (h) If $f(x, y, z)$ has an absolute minimum at $(0, 0, 0)$, and S is the surface $z = x^2 + y^2$, then f constrained to S also has an absolute minimum at the origin. __________

   (i) For any smooth planar vector &eld $\mathbf{F}$, with $\text{curl}(\mathbf{F}) = \mathbf{0}$, there exists a scalar function $\phi(x, y)$ with

   $$\mathbf{F} = \text{grad}(\phi).$$

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   (j) For any smooth vector &eld $\mathbf{F}(x, y, z)$,

   $$\text{grad}(\text{div}(\mathbf{F})) = \mathbf{0}.$$

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2. For the curve $\mathbf{c}(t) = \langle t, \sqrt{6}t^2/2, t^3 \rangle$ &nd [10]

   (a) The arc length between $t = 1$ and $t = 2$.

   (b) The curvature κ.

3. The hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ is called a ruled surface, which is to say that it is made up of straight lines. Prove this by showing that for any θ, the line [10]

$$\frac{x - \cos\theta}{\sin\theta} = \frac{y - \sin\theta}{-\cos\theta} = \frac{z}{1}$$

lies entirely in the surface.

4. Show that any smooth function z of the form [10]

$$z = f(x - ct) + g(x + ct)$$

satis&es the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2}.$$

5. Show that the product of the x, y, and z intercepts of any tangent plane to the surface $xyz = 1$ in the &rst octant is a constant. [10]

6. A rectangular box without a lid is to be made from $12m^2$ of cardboard. [10]
Find the maximum volume of such a box.

7. Sketch the solid whose volume is given by the iterated integral [10]

$$\int_0^2 \int_0^{2-z} \int_0^{4-y^2} 1 \; dx \, dy \, dz.$$

8. Find the centre of mass of the planar lamina which lies in the region de&ned by the inequalities $x^2+y^2 \leq 4$ and $y \geq 0$, with constant density $\delta = 1$. [10]

9. Let R be the region between the cicles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ [10] that also lies in the upper half plane $y \geq 0$. Hence R is a semi-annular region. Let c be the boundary of R, oriented in the positive sense. Evaluate

$$\int_c y^2\,dx + 3xy\,dy.$$

Hint: Think Green and then polar.

10. Let c be the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above, and $\mathbf{F}$ the vector &eld [10]

$$\mathbf{F}(x, y, z) = x^2 z\mathbf{i} + xy^2\mathbf{j} + z^2\mathbf{k} = \langle x^2 z, xy^2, z^2 \rangle$$

Compute

$$\int_c \mathbf{F} \cdot d\mathbf{r}.$$