AMAT 309 MIDTERM EXAM SOLUTIONS

Useful Formulas

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}; \qquad \mathbf{N} = \mathbf{B} \times \mathbf{T}; \qquad \mathbf{T} = \mathbf{N} \times \mathbf{B}$$

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \qquad \frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B} \qquad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$$

$$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|} \qquad \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \qquad \tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot (\frac{d\mathbf{a}}{dt})}{|\mathbf{v} \times \mathbf{a}|^2}$$

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + v^2 \kappa \mathbf{N} \qquad \kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

1. (3) For what values of a does $v = x^3 + axy^2$ satisfy $v_{xx} + v_{yy} = 0$? We calculate $v_x = 3x^2 + ay^2$, $v_y = 2axy$, so

$$v_{xx} + v_{yy} = 6x + 2ax = 0 \iff a = -3.$$

2. (3) Find the unit tangent vector and the binormal to the curve $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos^2(t)\mathbf{j}$. We calculate $\mathbf{v}(t) = \mathbf{r}'(t) = \cos(t)\mathbf{i} - 2\cos(t)\sin(t)\mathbf{j}$, so

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\left[\cos^2(t) + 4\cos^2(t)\sin^2(t)\right]^{\frac{1}{2}}}\left[\cos(t)\mathbf{i} - 2\cos(t)\sin(t)\mathbf{j}\right] =$$

$$= \frac{1}{\sqrt{1 + 4\sin^2(t)}}\left[\mathbf{i} - 2\sin(t)\mathbf{j}\right].$$

To find $\mathbf{B}$, we first calculate $\mathbf{a}(t) = \mathbf{r}''(t) = -\sin(t)\mathbf{i} - 2\cos(t)\mathbf{j}$, then

$$\mathbf{B}(t) = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$$

which is equal to the vector

$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & -2\cos(t)\sin(t) & 0 \\ -\sin(t) & -2\cos(t) & 0 \end{bmatrix}$$

divided by its magnitude. However, because of the zeros in the last column, the only nonzero component of this cross-product is the $\mathbf{k}$ component, and it is $-2\cos(t)\cos(2t) - \sin(t)\sin(2t)$. Since $\mathbf{B}$ has length 1, $B = -\mathbf{k}$. The minus sign comes from the two minus signs in that last formula, which tell us that a (-1) can be factored out before normalizing the vector to length 1–or equivalently, that the vector points DOWN the z-axis. (analagous to the fact that if you normalize the vector $-3\mathbf{k}$, you get $-\mathbf{k}$).

3. (3) Given the function $f(x, y, z) = ye^{zx^2}$, find (a) the gradient of f, (b) the directional derivative of this function at the point $(3, 2, 0)$ in the direction of the vector from that point to the point $(2, 3, 1)$.

   Solution: (a)

$$\vec{\nabla} f = f_x\mathbf{i} + f_y\mathbf{j} + f_{zz}\mathbf{k} = 2xzye^{zx^2}\mathbf{i} + e^{zx^2}\mathbf{j} + x^2ye^{zx^2}\mathbf{k}.$$

   (b) $\vec{\nabla} f\ (3, 2, 0) = 1\mathbf{j} + 18\mathbf{k},$. The direction we want is $(2-3)\mathbf{i} + (3-2)\mathbf{j} + (1-0)\mathbf{k}$ and we normalize to get

$$\vec{u} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}.$$

   Then

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = [1\mathbf{j} + 18\mathbf{k}] \cdot \left[-\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right] = \frac{19}{\sqrt{3}}.$$

4. (4) Find the arclength of the curve

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \ln(t)\mathbf{k}, \qquad 1 \le t \le 10.$$

   Solution. Arclength is the integral of speed:

$$s = \int_1^{10} \left|\vec{r'}(t)\right| dt = \int_1^{10} \left[2^2 + (2t)^2 + \left(\frac{1}{t}\right)^2\right]^{\frac{1}{2}} dt =$$

$$= \int_1^{10} \frac{1+2t^2}{t}\, dt = \left[\ln(t) + t^2\right]_1^{10} = \ln(10) + 99.$$

5. (4) Find the equation of the tangent plane to the surface $z = \sin(xy)$ at the point $(1, \pi, 0)$.

   Solution: A normal to the surface $z = f(x, y)$ is $\vec{N} = f_x\mathbf{i} + f_y\mathbf{j} + (-1)\mathbf{k}$, and evaluated at the given point this is

$$\vec{N} = y\cos(xy)\mathbf{i} + x\cos(xy)\mathbf{j} - \mathbf{k} = -\pi\mathbf{i} - 1\mathbf{j} - 1\mathbf{k}.$$

   Then the equation of the plane through the point $(1, \pi, 0)$ with this normal is

$$-\pi(x-1) - 1(y-\pi) + (-1)(z-0) = 0.$$

6. (3) Given the function $f(x, yz) = e^{xyz}$, find

$$\frac{\partial^3 f}{\partial x \partial y \partial z}.$$

   Solution: We calculate:

$$f_x = yze^{xyz}, \qquad f_{xy} = (f_x)_y = \left[z + xyz^2\right]e^{xyz}, \qquad f_{xyz} = (f_{xy})_z =$$

$$= \left[1 + 2xyz + xyz + x^2y^2z^2\right]e^{xyz} = \left[1 + 3xyz + x^2y^2z^2\right]e^{xyz}.$$

7. (6) Given the formula $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$ for a curve in R^3, derive the formula for the curvature of a plane curve $y = y(x)$: $\kappa(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$.

Solution: Any plane curve $y = f(x)$ can be interpreted as a space curve, using x as the parameter:

$$\vec{r}\,(x) = x\mathbf{i} + f(x)\mathbf{j} + 0\mathbf{k}, \Rightarrow \vec{r}\,'(x) = 1\mathbf{i} + f'(x)\mathbf{j} + 0\mathbf{k}, \Rightarrow \vec{r}\,''(x) = 0\mathbf{i} + f''(x)\mathbf{j} + 0\mathbf{k}.$$

We plug these vectors into the given formula for the curvature of a space curve

$$\kappa(x) = \frac{1}{|\vec{r}\,'(x)|}\left|\vec{r}\,'(x)\times\vec{r}\,''(x)\right| = \frac{1}{\left[1+(f'(x))^2\right]^{\frac{1}{2}}}\left|\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & f'(x) & 0\\ 0 & f''(x) & 0\end{bmatrix}\right| = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{\frac{1}{2}}}.$$

8. (7) Let u, v be defined as functions of x and y by the equations

$$x = u^2 + v^2, \qquad y = u^3 - v^3,$$

in a neighborhood of the point $(x, y, u, v) = (2, 0, 1, 1) = P_0$.

(a) Show that the functions $u(x, y)$ and $v(x, y)$ exist near P_0.

(b) Find u_x and u_y at P_0.

(c) Write down the Taylor Polynomial of degree one at P_0 for the function $u(x, y)$.

Solution: (a) We can solve for u and v as functions of x and y near the given point if the Jacobian evaluated at that point is nonzero:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix}2u & 2v\\ 3u^2 & -3v^2\end{bmatrix} = \det\begin{bmatrix}2 & 2\\ 3 & -3\end{bmatrix} = -12.$$

(b) We differentiate the given equations with respect to x to get:

$$1 = 2uu_x + 2vv_x.\ 0 = 3u^2u_y - 3v^2v_y,$$

which we evaluate at P_0 to get

$$1 = 2u_x + 2v_x,\ 0 = 3u_x - 3v_x \ \Rightarrow u_x = \frac{1}{4},\ v_x = \frac{1}{4}.$$

To find u_y, we differentiate the original equations with respect to y and evaluate at P_0 to get:

$$0 = 2uu_y + 2vv_y,\ 1 = 3u^2u_y - 3v^2v_y \ \Rightarrow\ 0 = 2u_y + 2v_y,\ 1 = 3u_y - 3v_y \ \Rightarrow\ u_y = \frac{1}{6},\ v_y = -\frac{1}{6}.$$

Thus

$$u_x = \frac{1}{4} \qquad \text{and } u_y = \frac{1}{6}.$$

(c) Using these values of u_x and u_y, the Taylor Polynomial of degree one for u about P_0 is

$$u(P_0) + \vec{\nabla}u\,(P_0)\cdot[(x-2)\mathbf{i} + (y-0)\mathbf{j}] = \left[1 + \frac{1}{4}\mathbf{i} + \frac{1}{6}\mathbf{j}\right]\cdot[(x-2)\mathbf{i} + (y-0)\mathbf{j}] =$$

$= 1 + \frac{1}{4}(x-2) + \frac{1}{6}y.$

9. (5) Show that

$$\frac{\cos(x)}{\cos(y)} \approx 1 - \frac{1}{2}\left(x^2 + y^2\right)$$

for (x, y) near $(0, 0)$. (Hint: Taylor)

Solution: Let $f(x, y) = \frac{\cos(x)}{\cos(y)}$. We want to calculate the Taylor polynomial of degree two about $(0, 0)$, so we compute (at $(0, 0)$:

$$f_x = -\frac{\sin(x)}{\cos(y)} = 0, \; f_y = \frac{\cos(x)\sin(y)}{\cos^2(y)} = 0.$$

So the linear approximation is zero . Now we go to the quadratic terms:

$$f_{xx} = -\frac{\cos(x)}{\cos(y)} = -1, \; f_{xy} = -\frac{\sin(x)\sin(y)}{\cos^2(y)} = 0$$

$$f_{yy} = \cos(x)\left[\tan(y)\sec(y)\right]_y = \cos(x)\left[\sec^3(y) + \tan^2(y)\sec(y)\right] = 1.$$

Then the quadratic Taylor polynomial for f about $(0, 0)$ is

$$1 + 0 \cdot (x - 0) + 0 \cdot (y - 0) + \frac{1}{2!}(-1) \cdot (x - 0)^2 + \frac{2}{2!}0 \cdot (x - 0)(y - 0) + \frac{1}{2!}(1) \cdot (y - 0)^2 =$$

$$= 1 - \frac{1}{2}\left[x^2 - y^2\right].$$

Alternate solution: We know from one dimensional calculus that $\cos(u) \approx 1 - \frac{1}{2!}u^2$, so

$$\frac{\cos(x)}{\cos(y)} \approx \frac{1 - \frac{1}{2!}x^2}{1 - \frac{1}{2!}y^2},$$

and we know from high school study of geometric sums that for $|r| < 1$

$$\frac{1}{1 - r} = 1 + r + r^2 + \cdots \approx 1 + r,$$

so

$$\frac{\cos(x)}{\cos(y)} \approx \left[1 - \frac{1}{2!}x^2\right]\left[1 + \frac{1}{2!}y^2\right],$$

and if we mulitply out, keeping only terms up to quadratic, we get the desired approximation.

10. (6) Locate and classify the critical points of $f(x, y) = xye^{x+y}$. Some help:

$$f_{xx} = y(2 + x)e^{x+y}, \;\; f_{xy} = (1 + x)(1 + y)e^{x+y}, \;\; f_{yy} = x(2 + y)e^{x+y}.$$

Solution: We calculate the first partials:

$$f_x = y(1 + x)e^{(x+y)}, \qquad f_y = x(1 + y)e^{(x+y)}.$$

These both vanish at the points $(0, 0)$ and $(-1, -1)$. We form the Hessian at each point. At $(0, 0)$ we have

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and $\det(H) = -1$. Since $\det(H)$ is the product of the two eigenvalues of H, they differ in sign and so $(0,0)$ is a saddle.
At the point $(-1,-1)$ we calculate

$$\begin{bmatrix} -e^{-2} & 0 \\ 0 & -e^{-2} \end{bmatrix},$$

and this is a diagonal matrix with its eigenvalues displayed on the diagonal. Since both are negative, $(-1,-1)$ is a maximum.

11. (6) Let

$$f(x,t) = \int_0^{m(x,t)} e^{-u^2}\, du, \text{where } m(x,t) = \frac{x}{2\sqrt{kt}}.$$

Show that

(a)

$$f_x = e^{-m^2(x,t)} \frac{\partial m}{\partial x}, \quad f_t = e^{-m^2(x,t)} \frac{\partial m}{\partial t}$$

(b)

$$k\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$$

(the heat equation)

Solution: (a) We want to use the Fundamental Theorem of Calculus to deal with the derivative of the integral. To be very clear what's going on here, we define a function of one variable F by

$$F(q) = \int_0^q e^{-u^2}\, du.$$

Then $F'(q) = \frac{dF}{dq} = e^{-q^2}$ by the Fundamental Theorem of Calculus, and $f(x,t) = F\big(m(x,t)\big)$.
Therefore, by the Chain Rule,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\Big(F\big(m(x,t)\big)\Big) = F'(m)\frac{\partial m}{\partial x} = e^{-m^2}\frac{\partial m}{\partial x},$$

which is what we're required to show. A similar calculation holds for $\frac{\partial f}{\partial t}$.
(b) We finish the calculation of f_t started in (a):

$$f_t = e^{-m^2(x,t)}\frac{\partial m}{\partial t} = -e^{-m^2(x,t)}\frac{x}{4\sqrt{k}\, t^{\frac{3}{2}}}.$$

Now we differentiate f_x with respect to x:

$$f_{xx} = \left\{-2m(t,x)\left[\frac{\partial m}{\partial x}\right]^2 + \frac{\partial^2 m}{\partial x^2}\right\} e^{-m^2(x,t)} = \left[-\frac{x}{\sqrt{kt}}\left(\frac{1}{2\sqrt{kt}}\right)^2 + 0\right] e^{-m^2(x,t)}.$$

Comparing $k\frac{\partial^2 f}{\partial x^2}$ to $\frac{\partial f}{\partial t}$ we see they are equal.