Solutions to AMAT 309 Quiz 3   March, 2006   M. Hamilton, J. Macki

1. Find and classify all critical points of $f(x,y) = 2x^4 - xy^2 + 2y^2$.

   We compute $f_x = 8x^3 - y^2$, $f_y = -2xy + 4y$, and set both equal to zero to find the critical points:
$$8x^3 = y^2, \qquad 2y[2-x] = 0.$$
   This pair of equations has the solutions $x = 2,\ y = \pm 8$ and $y = 0,\ x = 0$.
   To classify, these points we first need the second partials:
$$f_{xx} = 24x^2, \qquad f_{xy} = -2y, \qquad f_{yy} - 2x + 4.$$
   At $0,0)$, the Hessian is
$$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix},$$
   with eigenvalues 0 and 4, so we have no information.

   At $(2,-8)$, we have
$$\begin{bmatrix} 96 & 16 \\ 16 & 0 \end{bmatrix},$$
   and the determinant is negative, so it is a saddle. At $(2,8)$, we have
$$\begin{bmatrix} 96 & -16 \\ -16 & 4 \end{bmatrix},$$
   which also has negative determinant, so it is a saddle.

2. Find all extrema of $f(x,y) = x^3 + y^3$, subject to the constraint $x + y = 2$. Restrict your search to $0 \le x$ and $0 \le y$.

   We use Lagrange multipliers:
$$3x^2 = \lambda, \qquad 3y^2 = \lambda, \qquad x + y = 2.$$
   Then $x^2 = y^2$, so $x = \pm y$, since $x + y = 2$ we must have $x = y$. Thus $x = 1,\ y = 1$. At this point $f(1,1) = 2$. (This is in fact a minimum, the max is attained at the boundary point $(0,2)$ and $(2,0)$, and is not discovered by Lagrange.)

3. Find the shortest distance from the origin to the surface $x^2y^2z = 1$.

   We are trying to minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x^2y^2z = 1$. So we use Lagrange:
$$2x = 2\lambda xy^2z, \qquad 2y = 2\lambda x^2yz, \qquad 2z = \lambda x^2y^2, \qquad x^2y^2z = 1.$$
   If we multiply equation 1 by x, equation 2 by y, and the third equation by z, in each case the last equation (the constraint) can be used to reduce the new right hand side to λ:
$$x^2 = \lambda, \qquad y^2 = \lambda, \qquad 2z^2 = \lambda, \qquad x^2y^2z = 1.$$

If we use the right side of the first three equations to form the product x^2y^2z, we conclude that

$$\lambda\lambda\frac{\sqrt{\lambda}}{\sqrt{2}} = 1 \;\Rightarrow\; \lambda = 2^{\frac{1}{5}} \;\Rightarrow$$

$$\Rightarrow x = \pm 2^{\frac{1}{10}}, \qquad y = \pm 2^{\frac{1}{10}}, \qquad z = 2^{-\frac{2}{5}}.$$

(z must be positive by the constraint equation). So we have the critical points

$$(2^{\frac{1}{10}}, 2^{\frac{1}{10}}, 2^{-\frac{2}{5}}), \quad (-2^{\frac{1}{10}}, 2^{\frac{1}{10}}, 2^{-\frac{2}{5}}), \quad (2^{\frac{1}{10}}, -2^{\frac{1}{10}}, 2^{-\frac{2}{5}}) \quad (-2^{\frac{1}{10}}, -2^{\frac{1}{10}}, 2^{-\frac{2}{5}}).$$

The distance function has the same value at each of these points,

$$f = 2^{\frac{1}{5}} + 2^{\frac{1}{5}} + 2^{-\frac{4}{5}} = 2^{\frac{6}{5}} - 2^{-\frac{4}{5}} = \frac{3}{2^{\frac{4}{5}}}.$$