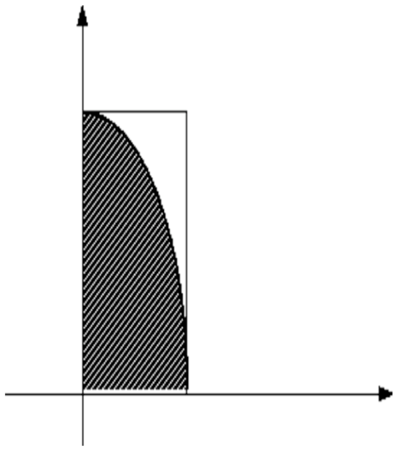
AMAT 309 Quiz 4 Solutions March, 2006 M. Hamilton, J. Macki

1. Reverse the order of integration and calculate one of the iterated integrals (your choice):

$$\int_0^2 dx \int_0^{4-x^2} xy\, dy.$$

Solution: A sketch with the function $y = 4 - x^2$ shows that y is going from the x-axis to the parabola, and x is going from 0 to 2.



The reverse order gives

$$\int_0^4 dy \int_0^{\sqrt{4-y}} xy\, dx.$$

This last iterated integral is calculated as follows:

$$\int_0^4 y \left[\frac{1}{2}(4-y)\right] dy = \left[y^2 - \frac{1}{6}y^3\right] |_0^4 = \frac{16}{3}.$$

2. Find the volume of the solid bounded by the xy-plane, the cylinder $x^2+y^2 = 4$, and the plane $z = 2-y$.

Solution: If D is the disc of radius 2 centred on the origin, the the volume is (using polar coordinates in the x, y plane):

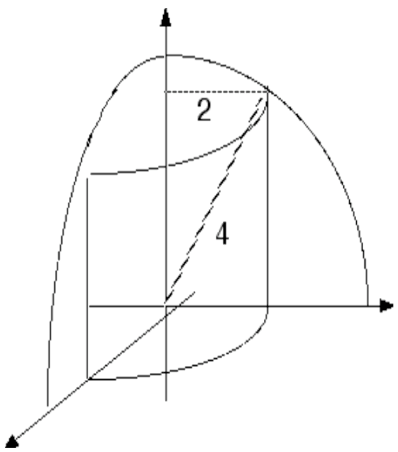
$$\iint_D dx\, dy \int_0^{2-y} dz = \iint_D (2-y)\, dx\, dy = \int_0^{2\pi} d\theta \int_0^2 (2 - r\sin(\theta))\, r\, dr =$$

$$= \int_0^{2\pi} \left[(r^2 - \frac{1}{3}r^3 \sin(\theta))\right] |_0^2\, d\theta = \int_0^{2\pi} \left[4 - \frac{8}{3}\sin(\theta)\right] d\theta = 8\pi.$$

3. Set up, but DO NOT EVALUATE, the triple integral for the volume inside the cylinder $x^2 + y^2 = 4$, bounded below by the xy-plane, and bounded above by the sphere $x^2 + y^2 + z^2 = 16$, in

(a) rectangular, (b) spherical and (c) cylindrical coordinates.

Some hints: Rectangular is not too bad, and cylindrical is easy. Spherical is hard. For spherical it may help to know that the boundary of the cylinder is defined by $\rho \sin(\phi) = 2$, and the intersection of the sphere with the cylinder is the circle $(z =)\ \rho\cos(\phi) = 2\sqrt{3}$, $(x^2 + y^2 =)\rho\sin(\phi) = 2$. Sketch the cylinder and sphere! You will get a sum of two integrals.



Solution: (a)

$$\int_0^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_0^{\sqrt{16-x^2-y^2}} dz,$$

$$\text{(b)} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} d\phi \int_0^4 \rho^2 \sin(\phi)\, d\rho + \int_0^{2\pi} d\theta \int_{\frac{\pi}{6}}^{\pi} d\phi \int_0^{\frac{2}{\sin(\phi)}} \rho^2 \sin(\phi)\, d\rho$$

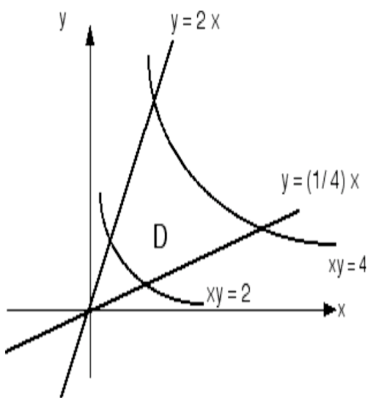
$$\text{(c)} \int_0^{2\pi} d\theta \int_0^2 dr \int\int_0^{\sqrt{16-r^2}} r dz$$

4. Use the change of variables $u = xy$ and $v = \frac{y}{x}$ to evaluate the double integral

$$\int_D\int \frac{y}{x}\sin(xy)dxdy.$$

Note: The Jacobian determinant satisfies the simple relation

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}.$$



Solution: We calculate the Jacobian determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2\frac{y}{x},$$

so $\frac{\partial(x,y)}{\partial(u,v)} = \frac{x}{2y} = \frac{1}{2v}$, and our integral becomes, over the box

$$B = \{(u,v)|2 \le u \le 4, \qquad \frac{1}{4} \le v \le 2\},$$

$$\int\int_B [v\sin(u)]\frac{1}{2v}\, dudv = \frac{1}{2}\int_{\frac{1}{4}}^2 dv \int_2^4 \sin(u)\, du = \frac{7}{8}[\cos(2) - \cos(4)].$$