AMAT 309 Quiz 2 Solutions   Winter Term, 2006   M. Hamilton, J. Macki

1. (10 points) Do ONE of (a) or (b):
(a) Find the equation of the tangent plane to the surface

$$z = \arctan\left(\frac{y}{x}\right) \quad \text{at} \quad \left(1, 1, \frac{\pi}{4}\right).$$

Recall from your childhood:

$$(\arctan(u))' = \frac{1}{1+u^2}, \quad \tan(\pi/4) = 1, \quad \sec(\pi/4) = \sqrt{2}, \; 1 + \sec^2(u) = \tan^2(u).$$

Solution: We calculate the first partial derivatives:

$$z_x = -\frac{y}{x^2+y^2}, \quad z_y = \frac{x}{x^2+y^2}.$$

At the point $\left(1, 1, \frac{\pi}{4}\right)$ we calculate them:

$$z_x = -\frac{1}{2}, \quad z_y = \frac{1}{2}.$$

Then the equation of the tangent plane at the given point is

$$-\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - \left(z - \frac{\pi}{4}\right) = 0.$$

OR

(b) Show that $u(x, y, t) = \sin(kx)\sin(ky)\sin(\sqrt{2}\,akt)$ satisfies the two-dimensional wave equation

$$u_{xx} + u_{yy} = \frac{1}{a^2}\,u_{tt}.$$

Solution: We calculate the second partial derivatives of u with respect to x, y and t:

$$u_{xx} = -k^2 u, \quad u_{yy} = -k^2 u, \quad u_{tt} = -2a^2k^2 u,$$

and the result immediately follows.

2. (15 points) Find $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$, τ, κ, and ρ at the point $(0, 1, 0)$ on the curve

$$\vec{r}(t) = \sin(t)\,\hat{\imath} \ + \ \cos(t)\,\hat{\jmath} \ + \ \tan(t)\,\hat{k}, \quad -\frac{\pi}{4} \le t \le \frac{\pi}{4}.$$

Solution: We calculate

$$\vec{r}\,'(t) = \cos(t)\,\hat{\imath} \ - \ \sin(t)\,\hat{\jmath} \ + \ \sec^2(t)\,\hat{k},$$

$$\vec{r}\,''(t) = -\sin(t)\,\hat{\imath} \ - \ \cos(t)\,\hat{\jmath} \ + 2\sec^2(t)\tan(t)\,\hat{k},$$

$$\vec{r}\,'''(t) = -\cos\,\hat{\imath} \ + \ \sin(t)\,\hat{\jmath} \ + \left[4\sec^2(t)\tan(t) + 2\sec^4(t)\right]\hat{k}.$$

We evaluate these at the point in question which corresponds to $t = 1$:

$$\vec{r} = \hat{\jmath}, \quad \vec{r}\,' = \hat{\imath} + \hat{k}, \quad \vec{r}\,'' = -\hat{\jmath}, \quad \vec{r}\,''' = -\hat{\imath} \ + 2\hat{k}.$$

Now we calculate: $\vec{v} \times \vec{a} = \hat{\imath} - \hat{k}$ so

$$\hat{T} = \frac{1}{\sqrt{2}}\left[\hat{\imath} + \hat{k}\right], \quad \hat{B} = \frac{1}{\sqrt{2}}\left[\hat{\imath} - \hat{k}\right].$$

Then $\hat{N} = \hat{B} \times \hat{T} = -\hat{\imath}$. Finally,

$$\kappa = \frac{1}{2}, \quad \rho = 2, \quad \tau = \frac{\left[\hat{\imath} - \hat{k}\right] \cdot \left[-\hat{\imath} \ + 2\hat{k}\right]}{\left(\sqrt{2}\right)^2} = -\frac{3}{2}.$$

Handy Formulas:

$$\hat{\mathbf{B}} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}, \quad \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}, \quad \tau = \frac{[\mathbf{v} \times \mathbf{a}] \cdot \frac{\mathbf{da}}{\mathbf{dt}}}{|\mathbf{v} \times \mathbf{a}|^2}.$$

Hint: Evaluate the vectors $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ at the given point before jumping into your calculations.