AMAT 309 Assignment 1 Winter Term, 2006

SOLUTIONS

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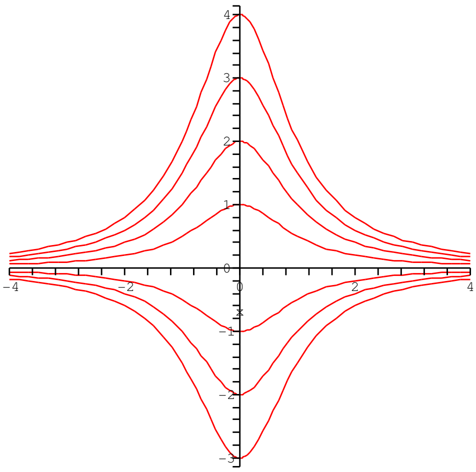
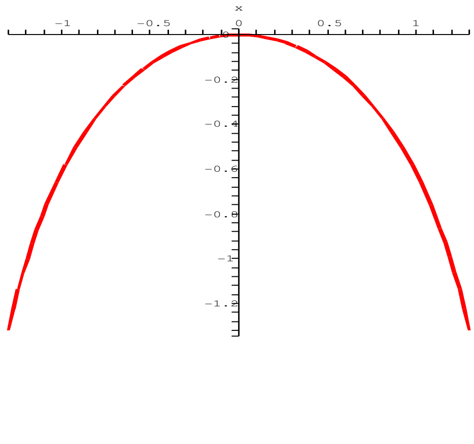
1. Sketch the level curves of the surface $z = y\,(x^2 + 1)$.

Solution: These will be the curves $y\,(x^2 + 1) = k$ in the $(x,\, y)$ plane. In the sketch the y-intercept value is k:



2. Sketch the graph of $y = \ln(\cos(x))$, $\;-\frac{\pi}{2} < x < \frac{\pi}{2}$. What is the equation of the osculating circle at $(0, 0)$?
What is $\kappa(x)$?

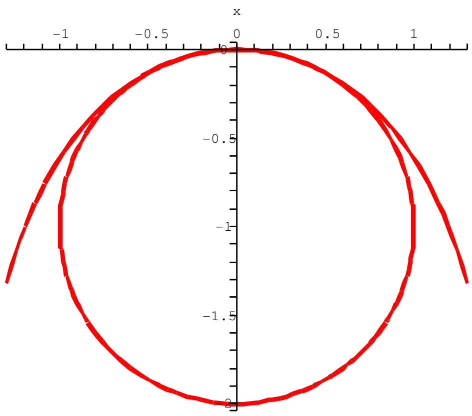
Here is the graph. Note there are vertical asymptotes at $\pm\frac{\pi}{2}$.



If $y = \ln(\cos(x))$, then $y' = -\tan(x)$ and $y'' = -\sec^2(x)$. Thus

$$\begin{aligned}\kappa(x) &= \frac{|y''|}{\left(1+(y')^2\right)^{\frac{3}{2}}} \\ &= \frac{|-\sec^2(x)|}{\left(1+\tan^2(x)\right)^{3/2}} \\ &= \frac{\sec^2(x)}{\sec^3(x)} \\ &= \cos(x)\end{aligned}$$

At $(0,0)$, $\kappa = 1$, and so $\rho = \frac{1}{\kappa} = 1$. The normal to the curve at $(0,0)$ is obviously $-\mathbf{j}$ (horizontal tangent, curving downwards), so the centre of the osculating circle is at $(0,-1)$ with radius 1, and so the equation of the circle is $x^2 + (y+1)^2 = 1$.



3. The formula

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right) = -1$$

is used in thermodynamics. Interpret this equation, and prove it. (Hint: Think implicit functions)

Solution: The confusing thing about this formula (at least the thing that confused me when I took thermodynamics) is that is assumes there is some specific relation among x, y, and z lurking in the background, which isn't made explicit.

So, assume we have some condition relating x, y, and z, which in general we can write as $F(x, y, z) = 0$. Then we can view any one of the variables x, y, z as functions of the other two (assuming, of course, that none of the partial derivatives of F are zero). What the formula says is that if we consider first $z(x, y)$ and take its partial derivative with respect to x (i.e. with y held constant), then consider $x(y, z)$ and take $\frac{\partial x}{\partial y}$, and then consider $y(x, z)$ and take $\frac{\partial y}{\partial z}$, then the product of these three terms is -1.

To prove it: if we differentiate the equation $F(x, y, z) = 0$ implicitly with respect to x, treating y as constant and z as a function of x and y, we get

$$F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial x} = 0$$

so that

$$\frac{\partial z}{\partial x} = -\frac{F_1(x, y, z)}{F_3(x, y, z)}.$$

Similar calculations give

$$\frac{\partial x}{\partial y} = -\frac{F_2(x, y, z)}{F_1(x, y, z)} \qquad \frac{\partial y}{\partial z} = -\frac{F_3(x, y, z)}{F_2(x, y, z)}.$$

Thus,

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right) = \left(-\frac{F_1}{F_3}\right)\left(-\frac{F_2}{F_1}\right)\left(-\frac{F_3}{F_2}\right) = -1.$$

(Aside: The generalization of this formula to more variables is interesting—see Adams, Exercise 12.8.25.)

4. Find the point(s) of maximum curvature on the graph of $y = \sinh x$.

Solution: The curvature of a plane curve $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{\big(1 + (f'(x))^2\big)^{3/2}}$$

which is

$$\kappa(x) = \frac{|\sinh(x)|}{\big(1 + \cosh^2(x)\big)^{3/2}} \tag{1}$$

in this case. We are looking for the maximum value of κ.

The absolute value is a bit of a problem, so we first consider $\sinh(x) > 0$, i.e. $x > 0$. Then

$$\kappa(x) = \frac{\sinh(x)}{\big(1 + \cosh^2(x)\big)^{3/2}}.$$

To find the maximum value, look for points where $\kappa'(x) = 0$. Differentiating, we get:

$$\kappa'(x) = \frac{\cosh(x)\big(1 + \cosh^2(x)\big)^{3/2} - \sinh(x)\frac{3}{2}\big(1 + \cosh^2\big)^{1/2} 2\sinh(x)\cosh(x)}{\big(1 + \cosh^2(x)\big)^3}$$

This will be zero iff the numerator is zero. Factor out a $\big(1 + \cosh^2\big)^{1/2}$ to get

$$\text{numerator} = \big(1 + \cosh^2\big)^{1/2}\Big(\cosh(x)\big(1 + \cosh^2(x)\big) - 3\sinh^2(x)\cosh(x)\Big) = 0$$

Now, $\big(1 + \cosh^2\big)^{1/2} \geq 1$ (and so is never zero), and so we seek roots of the other long term in brackets. Remembering that $\cosh^2(x) - \sinh^2(x) = 1$, we get

$$\begin{aligned} \cosh(x) + \cosh^3(x) - 3\big(\cosh^2(x) - 1\big)\cosh(x) &= 0 \\ \cosh(x) + \cosh^3(x) - 3\cosh^3(x) + 3\cosh(x) &= 0, \end{aligned}$$

i.e.

$$\begin{aligned} -2\cosh^3(x) + 4\cosh(x) &= 0 \\ \cosh(x)\big(2 - \cosh^2(x)\big) &= 0 \end{aligned}$$

Since $\cosh(x) \geq 1$, the only solution is $\cosh(x) = \sqrt{2}$.

Therefore $\sinh(x) = 1$, and so $x = \sinh^{-1}(1)$.

To find a value for x, we can use the fact that $\cosh(x) = \sqrt{2}$, so

$$e^x + e^{-x} = 2\sqrt{2}$$

and solve using the quadratic formula. We get that $x = \ln(1 + \sqrt{2})$, and

$$\kappa(x) = \frac{\sinh(x)}{\big(1 + \cosh^2(x)\big)^{3/2}} = \frac{1}{(1+2)^{3/2}} = \frac{1}{\sqrt{27}}.$$

There is only one critical point of $\kappa(x)$ for $x > 0$. Because κ is always positive, $\kappa(0) = 0$, and $\kappa \to 0$ as $x \to \infty$, we can conclude that this critical point is a maximum.

Next, we consider the case when $x < 0$. Since sinh is an odd function, the graph of sinh is symmetric about the origin. Thus, there will be another point with the same curvature which is the reflection of the above point, namely $\big(-\ln(1 + \sqrt{2}), -1\big)$. Another way to see this is that the expression (1) for κ is an even function, and so there will be corresponding maximum with x negative.

And finally, if $x = 0$, then $\kappa = 0$, so this is not a maximum (it is in fact a minimum, since $\kappa \geq 0$).

Therefore, the points of maximum curvature are $\big(\ln(1 + \sqrt{2}), 1\big)$ and $\big(-\ln(1 + \sqrt{2}), -1\big)$.

5. (a) Show that the surfaces $z = x^2y$ and $y = \frac{1}{4}x^2 + \frac{3}{4}$ intersect orthogonally at $(1, 1, 1)$.

(b) Show that the surfaces $z = f(x, y) = -x^2 - y^2$ and $z = g(x, y) = \frac{1}{4}\ln(xy)$ intersect orthogonally along their entire curve of intersection.

Solution: The angle between two surfaces is the angle between their tangent planes; the angle between two planes is the angle between their normal vectors. Thus, two surfaces are orthogonal if their normal vectors are orthogonal. If a surface is written as the level surface of a function of three variables (i.e. $F(x, y, z) = k$ for some function F and constant k), then the gradient vector of F is normal to the surface; thus, two level surfaces are orthogonal if their gradients are perpendicular.

(a) We can view these surfaces as the level curves $F = 0$ and $G = -\frac{3}{4}$ of the functions $F(x, y, z) = x^2y - z$ and $G(x, y, z) = \frac{1}{4}x^2 - y$, respectively.

$$\begin{aligned} \nabla F &= 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k} && \text{so } \nabla F(1,1,1) = 2\mathbf{i} + \mathbf{j} - \mathbf{k} \\ \nabla G &= \tfrac{1}{2}x\mathbf{i} - \mathbf{j} && \text{so } \nabla G(1,1,1) = \tfrac{1}{2}\mathbf{i} - \mathbf{j} \end{aligned}$$

Therefore

$$\nabla F \cdot \nabla G = 2 \cdot \tfrac{1}{2} + (-1) + 0 = 0$$

and so the two surfaces are orthogonal.

(b) Again, view these surfaces as level surfaces, of the functions $F(x, y, z) = z + x^2 + y^2$ and $G(x, y, z) = z - \frac{1}{4}\ln(xy)$. Then

$$\begin{aligned} \nabla F &= 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \qquad \text{and} \\ \nabla G &= -\tfrac{1}{4x}\mathbf{i} - \frac{1}{4y}\mathbf{j} + \mathbf{k}, \end{aligned}$$

so

$$\nabla F \cdot \nabla G = -\tfrac{2x}{4x} + -\tfrac{2y}{4y} + 1 = 0$$

for any x, y. Thus ∇F and ∇G are orthogonal at any point where the surfaces intersect.

6. (Problem 24, p. 742). Given a function $u(r, \theta)$ in polar coordinates, show that the Laplacian operator $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Solution: We are given $u(x, y)$ with $r^2 = x^2 + y^2$, $\tan(\theta) = \frac{y}{x}$, and we need to express $u_{xx} + u_{yy}$ in terms of partials of u with respect to r and θ.

$$r^2 = x^2 + y^2 \Rightarrow 2rr_x = 2x,\ 2rr_y = 2y.$$

$$\tan(\theta) = \frac{y}{x} \Rightarrow \sec^2(\theta)\theta_x = -\frac{y}{x^2}, \qquad \sec^2(\theta)\theta_y = \frac{1}{x}.$$

These simplify to give us

$$r_x = \cos(\theta), \quad r_y = \sin(\theta), \quad \theta_x = -\frac{\sin(\theta)}{r}, \quad \theta_y = \frac{\cos(\theta)}{r}.$$

We now differentiate these last expressions in x and y:

$$r_{xx} = -\sin(\theta)\theta_x = \frac{\sin^2(\theta)}{r}, \ r_{yy} = \cos(\theta)\theta_y = \frac{\cos^2(\theta)}{r},$$

$$\theta_{xx} = \frac{1}{r^2} r_x \sin(\theta) - \frac{\cos(\theta)}{r}\theta_x = \frac{2\cos(\theta)\sin(\theta)}{r^2}, \quad \theta_{yy} = -\frac{2\sin(\theta)}{r^2}.$$

Now we can go to work:

$$u_x = u_\theta \theta_x + u_r r_x,$$

$$u_{xx} = (u_\theta)_x \theta_x + u_\theta \theta_{xx} + (u_r)_x r_x + u_r r_{xx} =$$

$$(u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta \theta_{xx} + (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_r r_{xx} =$$

$$-2\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) u_{\theta r} + \left(\frac{\sin^2(\theta)}{r^2}\right) u_{\theta\theta} + 2\left(\frac{\cos(\theta)\sin(\theta)}{r^2}\right) u_\theta + \cos^2(\theta) u_{rr} + \left(\frac{\sin^2(\theta)}{r}\right) u_r.$$

The other half is to start with $u_y = u_\theta \theta_y + u_r r_y$:

$$u_{yy} = (u_\theta)_y \theta_y + u_\theta \theta_{yy} + (u_r)_y r_y + u_r r_{yy} =$$

$$= u_{\theta r} r_y \theta_y + u_{\theta\theta} (\theta_y)^2 + u_\theta \theta_{yy} + \big(u_{rr} (r_y)^2 + u_{r\theta} \theta_y r_y\big) + u_r r_{yy} =$$

$$= 2 \left(\frac{\sin(\theta)\cos(\theta)}{r}\right) u_{\theta r} + \left(\frac{\cos^2(\theta)}{r^2}\right) u_{\theta\theta} - 2 \left(\frac{\sin(\theta)\cos(\theta)}{r^2}\right) u_\theta + \sin^2(\theta) u_{rr} + \left(\frac{\cos^2(\theta)}{r}\right) u_r.$$

Putting these together, we get

$$u_{xx} + u_{yy} = \frac{1}{r^2} u_{\theta\theta} + u_{rr} + \frac{1}{r} u_r.$$

7. Given the system of equations $x = u^2 - uv,\ y = 3uv + 2v^2$:

(a) Show that this system can be solved for u and v as functions of x and y in a neighborhood of the point $(u, v, x, y) = (-1, 2, 3, 2)$.

(b) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ at this point.

(c) Use the linear approximation to estimate u and v for $(x, y) = (2.9, 2.02)$.

Solution: (a) Recall that, according to the Inverse Function Theorem, the system defines u and v as functions of x and y if the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero.

Thus,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u - v & -u \\ 3v & 3u + 4v \end{vmatrix}$$
$$= \begin{vmatrix} -4 & 1 \\ 6 & 5 \end{vmatrix}$$
$$= -26$$

This is not zero, and so u and v are defined as functions of x and y near the point $(-1, 2, 3, 2)$.

(b) Let J be the Jacobian matrix of u, v with respect to x, y at the point (-1,2,3,2). It will be the inverse matrix[1] of the Jacobian matrix of x, y with respect to u, v at this point, which is computed above. Thus

$$J = \begin{bmatrix} -4 & 1 \\ 6 & 5 \end{bmatrix}^{-1} = -\frac{1}{26}\begin{bmatrix} 5 & -1 \\ -6 & -4 \end{bmatrix}.$$

The entries in the first column of this matrix are u_x and v_x, and so

$$u_x = -\frac{5}{26}; \qquad v_x = \frac{6}{26} = \frac{3}{13}.$$

(c) The linear approximation for this situation is

$$\begin{bmatrix} u \\ v \end{bmatrix} \approx \begin{bmatrix} -1 \\ 2 \end{bmatrix} + J \cdot \begin{bmatrix} x - 3 \\ y - 2 \end{bmatrix};$$

here J is as above, $x = 2.9$, and $y = 2.02$. Thus

[1] Recall that the inverse of the 2×2 matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is $A^{-1} = \frac{1}{\det A}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$.

$$
\begin{aligned}
\begin{bmatrix} u \\ v \end{bmatrix} &\approx \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \frac{1}{26} \begin{bmatrix} 5 & -1 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} -0.1 \\ 0.02 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \frac{1}{26} \begin{bmatrix} -0.52 \\ 0.52 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.02 \\ -0.02 \end{bmatrix} \\
&= \begin{bmatrix} -0.98 \\ 1.98 \end{bmatrix}
\end{aligned}
$$

8. (This is Challenging Problem #1, p 781 in Adams)
(a) If the graph of a function $z = f(x, y)$ that is differentiable at (a, b) contains part of a straight line through $\big(a, b, f(a, b)\big)$, show that the line lies in the tangent plane to $z = f(x, y)$ at (a, b).

(b) If $g(t)$ is a differentiable function of t, describe the surface $z = y\, g(x/y)$ and show that all its tangent planes pass through the origin.

Solution: (a) This is easiest to do if we view the graph $z = f(x, y)$ as the 0-level surface of the function $F(x, y, z) = f(x, y) - z$, which we write for ease of notation as $F(\mathbf{r})$. Thus the surface is defined by the equation $F(\mathbf{r}) = 0$.

Let $\mathbf{r}_0 = \big(a, b, f(a, b)\big)$.

"The graph contains a straight line" means that some line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ lies in the surface, i.e. that $F(\mathbf{r}_0 + t\mathbf{v}) = 0$. We claim that $\mathbf{v} \perp \boldsymbol{\nabla} F$.

To see this, differentiate the equation $F(\mathbf{r}_0 + t\mathbf{v}) = 0$:

$$\begin{aligned} 0 = \frac{d}{dt}F(\mathbf{r}_0 + t\mathbf{v}) &= \boldsymbol{\nabla} F \cdot \frac{d}{dt}(\mathbf{r}_0 + t\mathbf{v}) \qquad \text{by the Chain Rule} \\ &= \boldsymbol{\nabla} F \cdot \mathbf{v} \end{aligned}$$

so $\boldsymbol{\nabla} F \perp \mathbf{v}$.

Alternatively, to do the same calculation explicitly in coordinates, note that the line has equation $(x, y, z) = \big(a + tv_1, b + tv_2, f(a, b) + tv_3\big)$, where $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Thus

$$0 = \frac{d}{dt}F\big(a + tv_1, b + tv_2, f(a, b) + tv_3\big) = \frac{\partial F}{\partial x}\, v_1 + \frac{\partial F}{\partial y}\, v_2 + \frac{\partial F}{\partial z}\, v_3$$

which is just $\boldsymbol{\nabla} F \cdot \mathbf{v}$.

Thus $\boldsymbol{\nabla} F \cdot \mathbf{v} = 0$. Now the tangent plane to the level surface $F(x, y, z) = 0$ at the point $\mathbf{r}_0$ is the plane passing through $\mathbf{r}_0$ perpendicular to $\boldsymbol{\nabla} F$. The line passes through $\mathbf{r}_0$ and its direction vector $\mathbf{v}$ is perpendicular to $\boldsymbol{\nabla} F$, and so therefore it lies in this plane.

(b) Imagine fixing y, and looking at the graph of $z = y\, g(x/y)$. This will be the graph of g, stretched horizontally and vertically by the same amount y. (If you don't remember this, review the transformations of graphs from first year or pre-calculus.) Thus cross-sections of this surface parallel to the x-axis will be stretched graphs of g, and the amount of the stretching will increase with y.

We can show the tangent planes all pass through the origin by a straightforward, brute-force calculation:

$$\frac{\partial z}{\partial y} = g\left(\tfrac{x}{y}\right) + y\, g'\left(\tfrac{x}{y}\right)\left(-\tfrac{x}{y^2}\right) = g\left(\tfrac{x}{y}\right) - \tfrac{x}{y}\, g'\left(\tfrac{x}{y}\right)$$

while

$$\frac{\partial z}{\partial x} = y\, g'\left(\tfrac{x}{y}\right)\left(\tfrac{1}{y}\right) = g'\left(\tfrac{x}{y}\right).$$

The tangent plane at (a, b) will be

$$\begin{aligned} z &= z(a, b) + \frac{\partial z}{\partial x}(x-a) + \frac{\partial z}{\partial y}(y-b) \\ &= b\, g\left(\tfrac{a}{b}\right) + \left[g\left(\tfrac{a}{b}\right) - \tfrac{a}{b}\, g'\left(\tfrac{a}{b}\right)\right](y-b) + g'\left(\tfrac{a}{b}\right)(x-a) \\ &= y\, g\left(\tfrac{a}{b}\right) - y\, \tfrac{a}{b} g'\left(\tfrac{a}{b}\right) + x\, g'\left(\tfrac{a}{b}\right) \end{aligned}$$

after expanding and simplifying. Thus, if $x = 0$ and $y = 0$, then $z = 0$ as well, and so the plane passes through the origin.

Alternatively, we can use part (a) as follows. Using the description of the surface $z = y\, g(x/y)$ given above, we see that as we move in the y-direction, the graph stretches horzontally and vertically by the same amount. Thus we expect that the graph will contain straight lines, radiating out from the origin.

We can make this precise: Suppose $y = mx$ for some m (and $x \neq 0$). What will be the value of z? Well,

$$z = y\, g\left(\tfrac{x}{y}\right) = mx\, g\left(\tfrac{x}{mx}\right) = mx\, g\left(\tfrac{1}{m}\right).$$

Therefore, the points $\left(x,\, mx,\, mx\, g(\tfrac{1}{m})\right)$ lie in the surface $z = y\, g(x/y)$, for all values of $x \neq 0$. This set of points is a straight line, which happens to pass through the origin. By part (a), this line must lie in the tangent plane, which must therefore pass through the origin.