1. Find and classify the extrema of $h(x, y) = \sin(x)\sin(y)\sin(x+y)$ on the square$[0,\pi] \times [0,\pi]$. (Keep in mind there is a boundary to check out).

Solution:

$$h_x = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y) = 0 \tag{1}$$
$$h_y = \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y) = 0 \tag{2}$$

Notice the second term in each equation is the same, so

$$\cos x \sin y \sin(x+y) = \sin x \cos y \sin(x+y). \tag{3}$$

Case I. $\sin y = \sin x = 0$ does not occur on the interior of $[0,\pi] \times [0,\pi]$, only on the boundary, which we'll deal with later.

Case II. $\cos x = 0 = \cos y \implies x = y = \pi/2$. In the original equations, this gives for (1) $-1 = 0$ and (2) $-1 = 0$ so it's no good.

Case III. $\sin(x+y) = 0$ implies $x + y = \pi$ (in this domain).
$y = \pi - x \implies \sin y = \sin x, \cos y = -\cos x$

$$(1)\ \ \cos x \sin x \cdot 0 + \sin^2 x(-1) = 0 \tag{4}$$
$$(2)\ \ -\sin x \cos x \cdot 0 + \sin^2 x(-1) = 0 \tag{5}$$

which together imply $x = 0$ or π, which is on the boundary.

So, with $\sin(x+y) \neq 0$, we can divide (3) to get

$$\cos x \sin y - \sin x \cos y = 0 \tag{6}$$
$$\sin(x-y) = 0 \quad \implies \quad x - y = 0 \text{ in interior of square} \tag{7}$$
$$\text{i.e. } x = y \tag{8}$$

Thus from (1) and (2),

$$\cos x \sin x \sin 2x + \sin x \sin x \cos 2x = 0$$

We are assuming here that $\sin x \neq 0$ (the case $\sin x = 0$ was dealt with in Case I), and so we can divide it out to get

$$\cos x \sin 2x + \sin x \cos 2x = 0$$

i.e. $\sin 3x = 0$, so $x = \pi/3$ or $2\pi/3$. So our critical points are $P_1 = (\frac{\pi}{3}, \frac{\pi}{3})$ and $P_2 = (\frac{2\pi}{3}, \frac{2\pi}{3})$.

Plugging these points back into (1) and (2), and remembering our 1-2-$\sqrt{3}$ triangle, we see they both work.

Then

$$h(P_1) = \tfrac{\sqrt{3}}{2} \cdot \tfrac{\sqrt{3}}{2} \cdot \tfrac{\sqrt{3}}{2} = \tfrac{3\sqrt{3}}{8} \quad \text{MAX}$$
$$h(P_2) = \tfrac{\sqrt{3}}{2} \cdot \tfrac{\sqrt{3}}{2} \cdot \tfrac{-\sqrt{3}}{2} = -\tfrac{3\sqrt{3}}{8} \quad \text{MIN}$$

Also check the boundary: On the boundary, one of x or y is either 0 or π, so $h \equiv 0$ on the boundary. Therefore the max and min are at the points found above.

2. Find those points on the curve of intersection of the surfaces $x^2 - xy + y^2 - z^2 = 1$ and $x^2 + y^2 = 1$ which are nearest the origin. (Hint: what was Lagrange's first name?)

Solution: The function to be minimized here is the distance $\sqrt{x^2+y^2+z^2}$, or equivalently (by the usual trick) the square of the distance, which we'll call $f = x^2 + y^2 + z^2$. The two constraints are $g = x^2 - xy + y^2 - z^2 - 1 = 0$ and $h = x^2 + y^2 - 1 = 0$.

Then the Lagrangian function is $L(x, y, z, \lambda, \mu) = f + \lambda g + \mu h$; looking for a critical point of L yields the five (!) equations

$$2x = \lambda(2x - y) + \mu(2x) \tag{9}$$
$$2y = \lambda(-x + 2y) + \mu(2y) \tag{10}$$
$$2z = \lambda(-2z) + \mu(0) \tag{11}$$
$$x^2 + y^2 = 1 \tag{12}$$
$$x^2 - xy + y^2 - z^2 = 1 \tag{13}$$
$$\tag{14}$$

Note that (12) and (13) imply

$$-xy - z^2 = 0. \tag{15}$$

Now, (11) implies $\lambda = -1$ OR $z = 0$.

Case I: $z = 0$. Then (15) implies $xy = 0$. But $x^2 + y^2 = 1$, so possible points are

$$P_1 = (0, 1, 0), \quad P_2 = (0, -1, 0), \quad P_3 = (1, 0, 0), \quad P_4 = (-1, 0, 0)$$

Check P_1 in (9) and (10) and get

$$0 = -\lambda, \qquad 2 = 2\lambda + 2\mu$$

so $\lambda = 0$, $\mu = 1$ will work. For P_2, we get

$$0 = -\lambda, \qquad -2 = \lambda(-2) + \mu(-2)$$

which works as well. And P_3 and P_4 check out as well. Therefore P_1, P_2, P_3, and P_4 are candidates, each at distance 1 from the origin.
Case II: $\lambda = -1$. Then (9) and (10) become

$$(4 - 2\mu)x = y \tag{16}$$
$$(4 - 2\mu)y = x, \tag{17}$$

and we also have $-xy - z^2 = 0$, $x^2 + y^2 = 1$.

If either x or y is zero, then (16) or (17) implies that the other is zero, which contradicts them being on the circle. So assume neither one is zero. Then we can divide (16) and (17) to get

$$\frac{x}{y} = \frac{y}{x} \qquad \Longrightarrow \qquad x^2 = y^2.$$

Since $x^2 + y^2 = 1$, we have $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{\sqrt{2}}$. Since $z^2 = -xy$, we have four possibilities:

$$\left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right),$$
$$\left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right),$$

each of which has distance $\sqrt{\frac{3}{2}}$ from the origin.

Thus $P_1 - P_4$ found before are the closest points.

(Note: The reference to Lagrange's first name is U of Alberta convention for hints. Lagrange's actual first name is not important.)

3. Find the volume of the finite solid bounded by the surfaces

$$az = x^2 + y^2, \;\; x^2 + y^2 + z^2 = 2a^2.$$

Solution: The surfaces intersect in the curve $x^2 + y^2 = a^2$; $z = a$, as can be seen by solving the two equations simultaneously. Thus the volume is

$$V = 4\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_{(x^2+y^2)/a}^{\sqrt{2a^2-x^2-y^2}} 1 \, dz \, dx \, dy$$

(the 4 comes from symmetry, and for the rest, draw a picture... which I can't duplicate easily here!).

This is easier in cylindrical coordinates, where it can be written (remembering that dV in cylindrical coordinates is $r\,dr\,dz\,d\theta$)

$$\begin{aligned}
V &= 4\int_0^{\frac{\pi}{2}}\int_0^a\int_{\frac{r^2}{a}}^{\sqrt{2a^2-r^2}} r\,dz\,dr\,d\theta \\
&= 4\int_0^{\frac{\pi}{2}} d\theta \int_0^a \left(\sqrt{2a^2-r^2}-\tfrac{r^2}{a}\right) r\,dr \\
&= 4\cdot\tfrac{\pi}{2}\cdot\left[-(\tfrac{1}{3}\sqrt{2a^2-r^2})^{3/2}-\tfrac{r^4}{4a}\right]_0^a \\
&= 2\pi\left(-\tfrac{a^3}{3}+\tfrac{1}{3}2^{3/2}a^3-\tfrac{a^3}{4}\right) \\
&= \tfrac{2\pi a^3}{3}\left(2\sqrt{2}-\tfrac{7}{4}\right)
\end{aligned}$$

4. Set up the correct limits for both iterated integrals for $\iint f(x,y)\,dA$ over D if D is:

1. The parallelogram with sides
$$x=3,\ x=5,\ 3x-2y+4=0,\ 3x-2y+1+0.$$
2. The triangle with sides $y=0,\ y=x,\ y=4-x$.
3. The finite domain cut out by the curves $y=x^2,\ y=4-x^2$.
4. The region bounded by the curve $\frac{(x-2)^2}{4}+\frac{(y-3)^2}{9}=1$.

Solution: For this question you really need to draw diagrams, which I don't have on the computer, so I'll just give the answers...

(a) The intersection points are $(5,8)$, $(5,\frac{19}{2})$, $(3,5)$, and $(3,\frac{13}{2})$.

(i) $\displaystyle\int_5^{\frac{13}{2}} dy\int_3^{\frac{-1+2y}{3}} f\,dx+\int_{\frac{13}{2}}^8 dy\int_{\frac{-4+2y}{3}}^{\frac{-1+2y}{3}} f\,dx+\int_8^{\frac{19}{2}} dy\int_{\frac{-4+2y}{3}}^5 f\,dx$

(ii) $\displaystyle\int_3^5 dx\int_{\frac{3x+1}{2}}^{\frac{3x+4}{2}} f\,dy$

(b) (i) $\displaystyle\int_0^2 dy\int_y^{4-y} f\,dx$

(ii) $\displaystyle\int_0^2 dx\int_0^x f\,dy+\int_2^4 dx\int_0^{4-x} f\,dy$

(c) (i) $\int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{x^2}^{4-x^2} f\,dy$

(ii) $\int_0^2 dy \int_{-\sqrt{y}}^{\sqrt{y}} f\,dx + \int_2^4 dy \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f\,dx$

(d) (i) $\int_0^6 dy \int_{2-2\sqrt{1-(y-3)^2/9}}^{2+2\sqrt{1-(y-3)^2/9}} f\,dx$

(ii) $\int_0^4 dx \int_{3-3\sqrt{1-(x-2)^2/4}}^{3+3\sqrt{1-(x-2)^2/4}} f\,dy$

5. Transform the following to polar coordinates, and evaluate:

1.
$$\int_0^2 dx \int_0^{\sqrt{4-x^2}} \ln\left(1+x^2+y^2\right)\,dy,$$

2.
$$\iint \arctan\left(\frac{y}{x}\right)\,dx, dy$$

over the region defined by

$$1 \le x^2+y^2 \le 9, \quad \frac{x}{\sqrt{3}} \le y \le x\sqrt{3}.$$

Solution: (a) We're integrating over the part of the disc of radius 2 that lies in the first quadrant.

$$\begin{aligned}
\int_0^2 dx \int_0^{\sqrt{4-x^2}} \ln\left(1+x^2+y^2\right)\,dy, &= \int_0^{\frac{\pi}{2}} d\theta \int_0^2 \ln(1+r^2) r\,dr\,d\theta \\
&= \tfrac{\pi}{2}\cdot\tfrac{1}{2}\int_1^5 \ln(u)\,du \qquad \text{letting } u = 1+r^2 \\
&= \tfrac{\pi}{4}\left(u\ln u - u\right)\Big]_1^5 = \tfrac{\pi}{4}\left(5(\ln 5 - 1) + 1\right) \\
&= \tfrac{\pi}{4}\left(5\ln 5 - 4\right)
\end{aligned}$$

(b) Again, draw a diagram. The region (call it R) lies between the circles of radius 1 and 3, and between the lines $y = \frac{x}{\sqrt{3}}$ and $y = x\sqrt{3}$, which correspond to $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$, respectively. Thus

$$\begin{aligned}
\iint_R \arctan\left(\frac{y}{x}\right)\,dx\,dy &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_1^3 \theta r\,dr\,d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \theta\left[\tfrac{1}{2}r^2\right]_1^3 d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \theta\left(\tfrac{9}{2}-\tfrac{1}{2}\right)d\theta \\
&= 4\left(\tfrac{1}{2}\theta^2\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}\right) = 2\left(\left(\tfrac{\pi}{6}\right)^2 - \left(\tfrac{\pi}{3}\right)^2\right) \\
&= \tfrac{\pi^2}{6}
\end{aligned}$$

6. Change to cylindrical or spherical coordinates and evaluate:

   1. $$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz,$$

   2. The volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere with equation $\rho = 4\cos(\phi)$

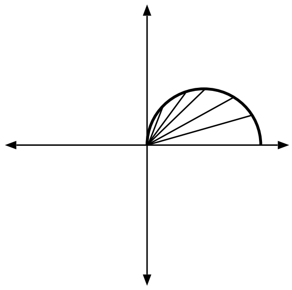
Solution: (a) The fact that we have a x^2+y^2 and a plain z appearing suggests that we use cylindrical coordinates. The z integral, therefore, is unchanged; what we need is to figure out the integral in x and y. The region in x and y over which we're integrating is $0 \le x \le 2$, $0 \le y \le \sqrt{2x-x^2}$. To find the curve $y = \sqrt{2x-x^2}$, complete the square:

$$\begin{aligned}
& y = \sqrt{2x-x^2} \\
\text{so} \quad & y^2 = 2x - x^2 = 1-(x-1)^2, \\
& \text{and so} \quad (x-1)^2 + y^2 = 1
\end{aligned}$$

This is the circle of radius 1, centred at $(1,0)$, which has polar equation $r = 2\cos\theta$. Since we're taking the positive square root, the region of integration is the top half of this circle (see figure below). (If you don't remember that equation in polar coordinates, take the second line of the above set of equations and convert to r and θ:

$$\begin{aligned}
r^2\sin^2\theta &= 2r\cos\theta - r^2\cos^2\theta \\
r^2 &= 2r\cos\theta \\
\text{so} \quad r = 0 \text{ or } r &= 2\cos\theta)
\end{aligned}$$

Thus the region is $0 \le r \le 2\cos\theta$, for $0 \le \theta \le \frac{\pi}{2}$. If it's not clear that this is the proper range of θ, consider the following diagram:



where some lines of constant θ have been marked.

Therefore, the integral becomes (remembering that $dz\,dy\,dx = r\,dz\,dr\,d\theta$)

$$\begin{aligned}
&\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^a z\sqrt{x^2+y^2}\,dz\,dy\,dx \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_0^a zr\,r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \tfrac{a^2}{2} r^2\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \tfrac{a^2}{6} r^3 \Big]_0^{2\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \tfrac{8a^2}{6} \cos^3\theta\,d\theta \\
&= \tfrac{4a^2}{3} \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\,d\theta = \tfrac{4a^2}{3} \int_0^{\frac{\pi}{2}} \cos\theta - \sin^2\theta\cos\theta\,d\theta \\
&= \tfrac{4a^2}{3}\big(\sin\theta - \tfrac{1}{3}\sin^3\theta\big)\Big]_0^{\frac{\pi}{2}} = \tfrac{4a^2}{3}\big(1-\tfrac{1}{3}\big) = \tfrac{8a^2}{9}
\end{aligned}$$

(b) The sphere $\rho = 4\cos\phi$ has radius 2 and sits "on top" of the z-axis (i.e. it has centre $(0,0,2)$). The surface $\phi = \frac{\pi}{3}$ is a cone. The volume element in spherical coordinates is $\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$, and so the volume is

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^{4\cos\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \\
&= 2\pi \int_0^{\frac{\pi}{3}} \tfrac{1}{3}\rho^3 \Big]_0^{4\cos\phi} d\phi \\
&= \tfrac{2\pi}{3} \int_0^{\frac{\pi}{3}} 64\cos^3\phi\,\sin\phi\,d\phi = \tfrac{2\pi}{3}\big(-\tfrac{64}{4}\cos^4\phi\big)\Big]_0^{\frac{\pi}{3}} \\
&= \tfrac{2\pi}{3}\tfrac{64}{4}\Big(1 - \big(\tfrac{1}{2}\big)^2\Big) = \tfrac{32\pi}{3} \cdot \tfrac{15}{16} = 10\pi.
\end{aligned}$$

7. Prove that the function

$$y(x) = \int_0^\infty \frac{e^{-xz}}{1+z^2}\,dz$$

satisfies the differential equation $y''(x) + y = \frac{1}{x}$.

Solution: If $y(x) = \displaystyle\int_0^\infty \frac{e^{-xz}}{1+z^2}\,dz$, then differentiating under the integral sign, we have

$$\begin{aligned} y'(x) &= \int_0^\infty \frac{\partial}{\partial x}\frac{e^{-xz}}{1+z^2}\,dz \\ &= \int_0^\infty \frac{-z\,e^{-xz}}{1+z^2}\,dz \\ \text{and} \qquad y''(x) &= \int_0^\infty \frac{z^2 e^{-xz}}{1+z^2}\,dz \end{aligned}$$

Thus

$$\begin{aligned} y'' + y' &= \int_0^\infty \frac{(1+z^2)e^{-xz}}{1+z^2}\,dz \\ &= \int_0^\infty e^{-xz}\,dz \\ &= \lim_{T\to\infty} \left(-\tfrac{1}{x}\right)e^{-xz}\Big]_0^T \\ &= \lim_{T\to\infty} \left(\tfrac{1}{x} - \tfrac{e^{-xT}}{x}\right) = \frac{1}{x} \end{aligned}$$

as required.

8. Given a region $D \subset \mathbb{R}^2$ in the plane and a function of two variables $f(x,y)$, let R be the region in space above D and below the graph of f. Then we have two expressions for the volume of R, namely

$$\iint_D f(x,y)\,dA \qquad \text{and} \qquad \iiint_R dV$$

Show that these two expressions are equal.

Solution: The region R can be described as $\{(x,y) \in D, 0 \le z \le f(x,y)\}$.

Therefore the triple integral over R can be written as an iterated integral:

$$\begin{aligned}\iiint_R dV &= \iint_D \int_0^{f(x,y)} dz\, dA \qquad \text{where } dA \text{ is the area element in the } xy \text{ plane} \\ &= \iint_D z\Big]_0^{f(x,y)} dA \\ &= \iint_D f(x,y)\, dA\end{aligned}$$

as required.

9. Let $f(x,y) = 3x^4 - 4x^2y + y^2$

1. Show that on each line $y = mx$, the function has a minimum at 0.
2. Show that $(0,0)$ is not a minimum of f.
3. Make a sketch (or use Maple) showing those points (x,y) where $f(x,y) > 0$ and $f(x,y) < 0$.

Hint: If you have trouble with part (b), perhaps try part (c) first.

Solution: (a) If $y = mx$, then

$$f(x,y) = f(x,mx) = 3x^4 - 4mx^3 + m^2x^2,$$

and we can treat this by the usual methods for functions of one variable. Differentiate:

$$f'() = 12x^3 - 12mx^2 + 2m^2x$$

and since $f' = 0$ at $x = 0$, there is a critical point at the origin. By the one-variable Second Derivative Test,

$$f''(0) = 36x^2 - 24mx + 2m\big|_{x=0} = 2m^2 > 0$$

and so the origin is a minimum along the line $y = mx$. (Note that the line $m = 0$ (i.e. $y \equiv 0$) also gives a minimum, since $f(x,0) = 3x^4$.) Also note that the value of f here is $f(0,0) = 0$.

(b) & (c) Following the hint, we consider points where $f > 0$ and $f < 0$.

Note that f factors as $f(x,y) = (3x^2 - y)(x^2 - y)$, and so $f(x,y) = 0$ whenever $y = 3x^2$ or $y = x^2$. Furthermore, f is negative if $x^2 < y < 3x^2$ (and positive elsewhere). Put another way, the function f is zero on the two curves $y = 3x^2$ and $y = x^2$, negative between them, and positive everywhere else.

Every neighbourhood of $(0, 0)$ contains points between the curves $y = x^2$ and $y = 3x^2$, and thus contains points where f is negative. Therefore the origin is not a miniumum of f.

Note: Many people said that the origin is not a minimum because the Hessian is zero. *This is not correct.* If the Hessian is zero, that means that the second derivative test gives us no information, which means just that: we have no information, i.e. we can't conclude anything. In particular, in this example we *cannot* conclude that $(0, 0)$ is not a minimum just because the Hessian is zero. We have to look more closely at the behaviour of the function.

(For example, the functions x^4+y^4, x^4-y^4, and $-x^4-y^4$ all have critical points at the origin where the Hessian is zero. However, they all have different behaviour there: minimum, saddle point, and maximum, respectively. Thus when the Hessian is zero, it means we need to do more analysis.)