Mathematics 309 Assignment 3

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1. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ upward through the surface

$$x(u,v) = u\cos(v), \quad y(u,v) = u\sin(v), \quad z(u,v) = u,$$

$0 \le u \le 2, \quad 0 \le v \le \pi.$

2. Consider $\int_C (xy + x + y)\,dx + (xy + x - y)\,dy$, where C is

$$\text{(a) the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{(b) the circle } x^2 + y^2 = ax.$$

Evaluate this integral (i) by calculating the line integral, and (ii) by using Green's Theorem. The curves are traversed in the positive (counterclockwise) direction. (*Hint:* For the integrals, you'll save yourself a lot of work if you pay attention to symmetry.)

3. Let $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\cos(1-x^2-y^2)\mathbf{j} + yz\mathbf{k}$. Find the flux of **curl F** upward through any surface whose boundary is the circle $x^2 + y^2 = 1,\ z = 2$.

4. Show that $\mathbf{F} = xe^{2z}\,\mathbf{i} + ye^{2z}\,\mathbf{j} - e^{2z}\,\mathbf{k}$ is a solenoidal vector field, and find a vector potential for it, i.e., a vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \mathbf{F}$.

5. Use the Divergence Theorem to evaluate the flux of the vector field

$$\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

across the boundary of the solid $x^2 + y^2 + 4(z-1)^2 \le 4$.
Note: To avoid spending excess time on the actual integration, do look for symmetries which will make an integral zero. In addition, it helps in some cases, with cylindrical coordinates, to integrate in the order z, θ, then r (the integral in θ can give you zero, eliminating the need to do a tough integral in r). Finally, if you get a tough integral, remember trig substitution and integrating even powers of trig functions–and use of tables (or Maple) to get the antiderivative is o.k. for this problem.

6. (This is Adams, p. 944, number 9) Let $\mathbf{F} = (x^2/y)\mathbf{i} + y\mathbf{j} + \mathbf{k}$.

   (a) Find the field line of $\mathbf{F}$ that passes through $(1, 1, 0)$ and show that it also passes through $(e, e, 1)$.

   (b) Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathcal{C}$ is the part of the field line in (a) from $(1, 1, 0)$ to $(e, e, 1)$.

7. Through what closed, oriented surface in $\mathbb{R}^3$ does the vector field

$$F = (4x + 2x^3 z)\mathbf{i} - y(x^2 + z^2)\mathbf{j} - (3x^2 z^2 + 4y^2 z)\mathbf{k}$$

   have the greatest flux?