AMAT 309 Quiz 5 **Solutions**

1. Given a wire in the shape of the curve $e^t\,\mathbf{i} + e^{2t}\,\mathbf{j}, \quad 0 \le t \le \pi$ in $\mathbb{R}^2$, with density $d = x$ at a point (x, y), find the mass of the wire.

Solution: Mass is density times length, so here the mass is $\int_C x\,ds$. For this curve, $\mathbf{r}' = e^t\,\mathbf{i} + 2e^{2t}\,\mathbf{j}$ and so $ds = |\mathbf{r}'| = \sqrt{e^{2t} + 4e^{4t}}\,dt$. Thus the mass is

$$\int_C x\,ds = \int_0^{\pi} e^t \sqrt{e^{2t} + 4e^{4t}}\,dt.$$

Let $u = e^t$, so the integral becomes

$$\int_1^{e^\pi} \sqrt{u^2 + 4u^4}\,du = \int_1^{e^\pi} u\sqrt{1 + 4u^2}\,du,$$

and another substitution ($v = 1 + 4u^2$) gives

$$\begin{aligned} &= \tfrac{1}{8} \cdot \tfrac{2}{3}\left(1 + 4u^2\right)^{\frac{3}{2}}\Big]_5^{e^\pi} \\ &= \tfrac{1}{12}\left(\left(1 + 4e^{2\pi}\right)^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) \end{aligned}$$

2. (a) Show that the vector field $\mathbf{F} = (y\cos x - \cos y)\,\mathbf{i} + (\sin x + x\sin y)\,\mathbf{j}$ is conservative by finding a potential ϕ for which $\nabla\phi = \mathbf{F}$.

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is any curve connecting $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, \pi)$.

Solution:

(a) If $\mathbf{F} = \nabla\phi$, then $\frac{\partial\phi}{\partial x} = y\cos x - \cos y$. This tells us that $\phi = y\sin x - x\cos y + g(y)$ for some function g of y only. Differentiating on y, we get $\frac{\partial\phi}{\partial y} = \sin x + x\sin y + g'(y)$, which tells us that g is a constant, which we may as well take to be zero. Thus if $\phi = y\sin x - x\cos y$, then $\mathbf{F} = \nabla\phi$ and thus is conservative.

(b) The line integral of a conservative vector field is the difference of the potential function between the end and the beginning of the curve. Thus

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \phi\left(\tfrac{3\pi}{2}, \pi\right) - \phi\left(0, \tfrac{\pi}{2}\right) \\ &= \left[y\sin x - x\cos y\right]_{(0,\frac{\pi}{2})}^{(\frac{3\pi}{2},\pi)} \\ &= \pi\sin\left(\tfrac{3\pi}{2}\right) - \tfrac{3\pi}{2}\cos(\pi) - \tfrac{\pi}{2}\sin(0) + 0\,\cos\left(\tfrac{\pi}{2}\right) \\ &= -\pi + \tfrac{3\pi}{2} - 0 + 0 = \tfrac{1}{2} \end{aligned}$$

3. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + 3z\,\mathbf{j} + xy^2\,\mathbf{k},$$

and C is the curve from $(1, 0, 0)$ to $(0, 1, \pi)$ defined by $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + 2t\,\mathbf{k}$.
(b) What is the value of the integral if we traverse the curve in the opposite direction?

Solution: (a) Here $\mathbf{r}'(t) = -\sin(t)\mathbf{i}+\cos(t)\mathbf{j}+2\mathbf{k}$, while $\mathbf{F}(\mathbf{r}) = 2\cos(t)\sin(t)\mathbf{i}+ 6t\mathbf{j} + \cos(t)\sin^2(t)\mathbf{k}$. Finally, $t = 0$ at the beginning and $t = \frac{\pi}{2}$ at the end. Thus

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} \langle 2\cos t \sin t\mathbf{i} + 6t\mathbf{j} + \cos t \sin^2 t\mathbf{k}\rangle \bullet \langle -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 2\,\mathbf{k}\rangle\, dt \\
&= \int_0^{\frac{\pi}{2}} -2\cos^2 t \sin t + 6t\cos t + 2\cos t \sin^2 t\, dt \\
&= \int_0^{\frac{\pi}{2}} 6t\cos t\, dt
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
&= 6t\sin t\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 6\sin t\, dt \\
&= 3\pi - 0 - \Big[-6\cos t\Big]_0^{\frac{\pi}{2}} \\
&= 3\pi - 6
\end{aligned}$$

(b) If we traverse the curve in the opposite direction, we get the negative of the line integral, i.e. $6 - 3\pi$.

4. Find the area of the surface $z = x^2 + y^2$, below the plane $z = 4$.

Solution: If we think of the graph of a surface $z = f(x, y)$ as being parametrized by x and y, then a normal vector is

$$\mathbf{N} = \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$dS = |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\tfrac{\partial z}{\partial x}\right)^2 + \left(\tfrac{\partial z}{\partial y}\right)^2 + 1}\, dx\, dy.$$

(Cf. Adams, Example 15.5.4, though don't worry about the geometry comments at the end.)

The surface area is $\iint_R dS$, where R is the region in the xy plane that corresponds to the surface—the projection of the surface onto the xy plane.

In this case, R is the disc $\{x^2 + y^2 \leq 4\}$, and the area is

$$\iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy.$$

This is easiest to evaluate in polar coordinates, where it becomes

$$\int_0^{2\pi} \int_0^2 r\sqrt{1 + 4r^2}\, dr\, d\theta,$$

which (after substituting for $1 + 4r^2$) gives

$$2\pi \cdot \tfrac{1}{8} \cdot \tfrac{2}{3}\big(1 + 4r^2\big)^{\frac{3}{2}}\Big]_0^2 = \tfrac{\pi}{6}\big(17\sqrt{17} - 1\big).$$