APPLIED MATHEMATICS 311 ASSIGNMENT #4

DUE DATE: DECEMBER 8, 2005

1. Obtain $\mathscr{L}\{\sin \omega t\}$ by (a) direct evaluation; (b) using the fact that $\sin \omega t$ satisfies the differential equation $Y'' + \omega^2 Y = 0$.

2. Draw a graph of the function

$$S(t) := H(t) + \sum_{n=1}^{\infty} (-1)^n H(t - na)$$

and hence justify thinking of S as a square wave. Here $a > 0$.

3. Consider the sequence of functions $\phi_n(t) = n/\pi(1+n^2t^2)$, $n = 1, 2, 3, \ldots$.

   (a) Show that

$$\int_{-\infty}^{+\infty} \phi_n(t)\, dt = 1.$$

   (b) Assuming that $f(t)$ is continuous at $t = 0$, show that

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \phi_n(t) f(t)\, dt = f(0).$$

   (c) Discuss a possible connection between the functions $\phi_n(t)$ and the delta function.

4. Find the inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{s}{(s-1)^4}\right\}.$$

5. Solve the integral equation

$$\int_0^t Y(u) \sin(t - u)\, du = Y(t) + \sin t - \cos t.$$

6. Show that

$$I = \int_0^{\infty} \frac{\sin tx}{x}\, dx = \frac{\pi}{2} \quad \text{if } t > 0.$$

(Hint: First show that

$$\mathscr{L}\{I\} = \int_0^{\infty} \frac{\mathscr{L}\{\sin tx\}}{x}\, dx$$

and evaluate the last integral.)

7. Let $P(s)$ and $Q(s)$ be polynomials in s where the degree of $P(s)$ is less than the degree of $Q(s)$ and where $Q(s) = 0$ has distinct roots $a_1, a_2, \ldots, a_n$. Prove *Heaviside's expansion formula*

$$\mathscr{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} e^{a_k t}\frac{P(a_k)}{Q'(a_k)}.$$

8. (Extra for experts). Generalize the previous exercise to the case where the roots may not be distinct and illustrate with an example.

9. (Extra for experts). Prove that

$$\mathscr{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\} = \frac{1-e^{-t}}{t}.$$