Winter '04.
Apr. 07

AMAT 411
TEST 2

Marks

1. Solve the inhomogeneous system $y' = Ay + \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$, $y(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, given the fundamental matrix solution is

$$Y(t) = e^{At} = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ e^t - e^{2t} & e^{2t} + te^t - e^t & e^{2t} \end{pmatrix}.$$

2. Find all critical points of the system

$$x' = x - 2xy$$
$$y' = -y + xy$$

and determine if they are stable, asymptotically stable or unstable.

3. Use the method of Lyapunov function to determine the stability of the origin $(0,0)$ of the system

$$x' = -x - \frac{x^3}{3}$$
$$y' = -y - y^3.$$

4. Find the eigenvalues and eigenfunctions of the Sturm-Liouville equation:

$$y'' + \lambda y = 0,$$
$$y(0) = 0, \quad y'(1) = 0.$$

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AMAT 411 (L01)
TEST 2 Solution

1. The solution is $y(t) = Y(t)\begin{pmatrix}1\\1\\0\end{pmatrix} + Y(t)\int_0^t Y^{-1}(s) f(s)\, ds$, where

$$Y^{-1}(t) = e^{-At} = \begin{pmatrix} e^{-t} & -te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ e^{-t}-e^{-2t} & e^{-2t}-(t+1)e^{-t} & e^{-2t} \end{pmatrix}, \quad f(s) = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$\int_0^t \begin{pmatrix} te^{-t} \\ 0 \\ t(e^{-t}-e^{-2t}) + te^{-2t} \end{pmatrix} dt$$

$$= \int_0^t \begin{pmatrix} te^{-t} \\ 0 \\ te^{-t} \end{pmatrix} dt = \begin{pmatrix} -e^{-t}(t+1) \\ 0 \\ -e^{-t}(t+1) \end{pmatrix}\Bigg]_0^t$$

$$= \begin{pmatrix} -e^{-t}(t+1) + 1 \\ 0 \\ -e^{-t}(t+1) + 1 \end{pmatrix}.$$

$\therefore$ The solution is

$$y(t) = Y(t)\begin{pmatrix} 2-e^{-t}(t+1) \\ 1 \\ 1-e^{-t}(t+1) \end{pmatrix}.$$

$$= \begin{pmatrix} 2e^{t} - (t+1) + te^{t} \\ e^{t} \\ (t+1)e^{t} - (t+1) \end{pmatrix}, \text{ after calculation.}$$

2. Critical points: $x(1-2y)=0$

$$-y(1-x)=0$$

$\therefore$ $x=0$, $y=\frac{1}{2}$

$y=0$, $x=1$

$\therefore$ 2 c.p. are $(0,0)$, $(1,\frac{1}{2})$

<u>Nature of $(0,0)$</u> $\quad x'=x$

$$y'=-y$$

The matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ has eigenvalues $\lambda=1,-1$.

$\therefore$ $(0,0)$ is unstable.

<u>Nature of $(1,\frac{1}{2})$</u> Let $u=x-1$.

$$v=y-\tfrac{1}{2}.$$

Then $u'=x'=u+1-2(u+1)(v+\frac{1}{2})=-2v-2uv$

$$v'=-(v+\tfrac{1}{2})+(u+1)(v+\tfrac{1}{2})=\frac{u}{2}+uv.$$

The matrix $\begin{pmatrix}0 & -2\\ \frac{1}{2} & 0\end{pmatrix}$ has e.v $\lambda=\pm i$

$\therefore$ $(1,\frac{1}{2})$ is stable.

3. Liapunov function $V=\frac{1}{2}(x^2+y^2)$

$\therefore$ $V'=xx'+yy'=x(-x-\frac{x^3}{3})+y(-y-y^3)$

$$=-(x^2+y^2+\frac{x^4}{3}+y^4)\leq 0 \quad (=0 \text{ at } (0,0))$$

$\therefore$ The origin $(0,0)$ is asymptotically stable.

4 $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$

$m^2 + \lambda = 0$

(i) If $\lambda = -\mu^2 < 0$ $(\mu > 0)$, then $m = \pm\mu$. and.

$y = c_1 e^{\mu t} + c_2 e^{-\mu t}$.

$y(0) = c_1 + c_2 = 0 \quad \therefore c_1 = -c_2$.

$y'(1) = \mu(c_1 e^{\mu} - c_2 e^{-\mu}) = 0 \quad \therefore \mu c_2 (+e^{\mu} + e^{-\mu}) = 0$

$\therefore c_2 = 0$ & $c_1 = 0$

$\therefore$ trivial solution

(ii). If $\lambda = 0$, then $y'' = 0$

$\therefore y = c_1 + c_2 t$.

$y(0) = c_1 = 0$ & $y(1) = c_2 = 0$

$\therefore$ trivial solution

(iii) If $\lambda = \mu^2 > 0$ $(\mu > 0)$, then $m = \pm\mu i$ and

$y = c_1 \cos \mu t + c_2 \sin \mu t$.

$y(0) = c_1 = 0$

$y'(1) = -\mu c_1 \sin\mu + \mu c_2 \cos\mu = 0$

$\therefore \cos\mu = 0$

$\therefore \mu = (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \cdots$

$\therefore \lambda = \mu^2 = (2n+1)^2 \frac{\pi^2}{4}, \quad$ ''

are the eigenvalues and the eigenfunctions are

$y = \sin (2n+1)\frac{\pi}{2} t, \quad n = 0, \pm 1, \pm 2, \cdots$