AMAT 411
TEST 1 (Solution)

1. Let $y = x^m(a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_0 a_n x^{n+m}$, $(a_0 \neq 0)$,
since $x=0$ is a regular singular pt.

Then

$$\sum_0 2a_n(n+m)(n+m-1)x^{n+m}$$

$$-\sum_0 a_n(n+m)x^{n+m} + \sum_0 a_n x^{n+m} + \sum_0 a_n x^{n+m} =$$

Equating to zero all coeff of powers of x:

$$x^m: \quad 2a_0 m(m-1) - a_0 m + a_0$$
$$= a_0 m(2m-2-1) + a_0 = a_0(2m^2 - 3m + 1)$$
$$= a_0(2m-1)(m-1) = 0$$
$$\therefore m = \tfrac{1}{2}, 1.$$

$$x^{n+m}: \quad a_n[(n+m)(2(n+m)-1)+1] = -a_{n-1}$$
$$a_n(2(n+m)-1)(n+m-1) = -a_{n-1}.$$

For $m = \frac{1}{2}$ $\quad a_1 = \dfrac{-a_0}{2\cdot\frac{1}{2}} = -a_0$

$$a_2 = \frac{-a_1}{4\cdot\frac{3}{2}} = -\frac{a_1}{6} = \frac{a_0}{6}.$$

One Soln is
$$y_1 = a_0 x^{\frac{1}{2}}\left(1 - x + \frac{x^2}{6} - \cdots\right).$$

For $m = 1$,

$$a_1 = -\frac{a_0}{3}$$

$$a_2 = \frac{-a_1}{5\cdot 2} = -\frac{a_1}{10} = \frac{a_0}{30}$$

The second solution is
$$y_2 = \tilde{a}_0 x\left(1 - \frac{x}{3} + \frac{x^2}{30} - \cdots\right)$$

The general solution is

$y = y_1 + y_2$

2. Try $y = ve^x$. $y' = e^x(v'+v)$, $y'' = e^x(v''+2v'+v)$.

$\therefore\ x\cancel{e^x}(v''+2v'+v) - (2x+1)\cancel{e^x}(v'+v) + (x+1)v\cancel{e^x} = 0$

$xv'' + v'(2x-2x-1) + v(x-2x-1+x+1) = 0$

$xv'' + v' = 0$

Let $p = v'$. Then $xp' - p = 0$ or $\frac{dp}{p} = \frac{dx}{x}$.

$\ln p = \ln x + \ln c_1$

$\therefore\ p = c_1 x$.

$\therefore\ v' = c_1 x$

$v = \int c_1 x\, dx = \frac{c_1}{2}x^2 + k_2$ or $k_1 x^2 + k_2$.

$(k_1 = \frac{c_1}{2})$.

$\therefore$ General solution is

$y = ve^x = e^x(k_1 x^2 + k_2)$

3. $y' = 2 + y^3$, $y(0) = 0$. Let $|x| \le a$, $|y| \le b$.

Then $|f| = |2+y^3| \le 2 + b^3 = M$

Since $f = 2 + y^3$ is cts and $f_y = 3y^2$ is bounded,
by the existence and uniqueness theorem, there exists
a unique solution $y(t)$, defined for

$$|x| \le h = \min\left(a, \frac{b}{M}\right)$$

Let $g(b) = \dfrac{b}{2+b^3}$.

To find max $g(b)$, we evaluate

$$g'(b) = \frac{2+b^3 - b \cdot 3b^2}{(2+b^3)^2} = \frac{2-2b^3}{(2+b^3)^2} = 0,$$

$$\therefore 2 = 2b^3.$$

$$\therefore b^3 = 1 \quad \therefore b = 1.$$

Since $g'(1^-) > 0$ and
$g'(1^+) > 0$,

$\therefore b = 1$ gives a rel. max. of $g(b)$

$$\therefore \text{Max } g(b) = \frac{1}{2+1} = \frac{1}{3},$$

Hence Solution exists for $|x| \le \frac{1}{3}$.

4\. (a) $y' = 1 + xy^2, \quad y(0) = 0.$

$\therefore y_n = \int_0^x 1 + x y_{n-1}^2 \, dx, \qquad n = 1, 2, \ldots, \; (y_0 = 0).$

$y_1 = \int_0^x 1 \, dx = x$

$y_2 = \int_0^x 1 + x^3 dx = x + \frac{x^4}{4}\Big]_0^x = x + \frac{x^4}{4}.$

$y_3 = \int_0^x 1 + x\left(x + \frac{x^4}{4}\right)^2 dx = x + \frac{x^4}{4} + \frac{x^7}{14} + \frac{x^{10}}{160}.$

(b) (i) $y' = y^{2/3}, \quad y(0) = 0$

One solution is $y \equiv 0$

The other solution is:

$\int_0^y \frac{dy}{y^{2/3}} = \int_0^x dx = x$ or $3y^{\frac{1}{3}} = x$. or

$y = \left(\frac{x}{3}\right)^3 = \frac{x^3}{27}.$

(ii) $f(y) = y^{2/3}$ is cts but f_y is not. Hence Lipschitz condition does not hold.