Winter 04 AMAT 411 #4 (Solution)

4/pg 218.

$$x' = 8x - y^2$$
$$y' = -y + x^2.$$

Critical pts: $\begin{cases} 8x - y^2 = 0 \\ -y + x^2 = 0 \end{cases} \Rightarrow \therefore 8x = y^2 = x^4$

$\therefore x(8 - x^3) = 0, \therefore x = 0, 2.$

$\therefore$ 2 c.p. $(0,0)$ & $(2,4)$.

Nature of C.p. (0,0)

$$x' = 8x + \varepsilon_1(x,y)$$
$$y' = -y + \varepsilon_2(x,y).$$

Here $\varepsilon_1 = -y^2$ & $\varepsilon_2 = x^2 \to o(r)$ as $r \to 0$.

$x^2+y^2 \geq y^2$
$\therefore \frac{1}{y^2} \geq \frac{1}{x^2+y^2}$
Similarly $\frac{1}{x^2} \geq \frac{1}{x^2+y^2}$
Or use polar coor.

$\left(\frac{|\varepsilon_1|}{r} = \frac{y^2}{(x^2+y^2)^{1/2}} \leq \frac{y^2}{(y^2)^{1/2}} = y \to 0 \text{ as } r \to 0, \right.$ i.e. $x \to 0, y \to 0$.

$\left. \frac{|\varepsilon_2|}{r} = \frac{x^2}{(x^2+y^2)^{1/2}} \leq \frac{x^2}{(x^2)^{1/2}} = x \to 0 \quad \text{''} \right)$

The nature of c.p. (0,0) can be deduced from the corr. linear system

$$x' = 8x$$
$$y' = -y.$$

The eigenvalues are $\lambda = -1, 8. \Rightarrow$ c.p. $(0,0)$ is an unstable saddle pt. (see fig. next page)

Nature of C.p. (2,4)

Put $\xi = x - 2$ and $\eta = y - 4$. (ie We convert c.p (2,4) to the origin (0,0).)

Then $\xi' = x' = 8(\xi+2) - (\eta+4)^2 = 8\xi - 8\eta - \eta^2.$
$\eta' = y' = -(\eta+4) + (\xi+2)^2 = 4\xi - \eta + \xi^2.$

Since, as above, η^2 & ξ^2 are $o(r)$, $r^2 = \eta^2 + \xi^2$, the nature of c.p. (2,4) is the same as that of (0,0) of the Corr. linear system:

$$\xi' = 8\xi - 8\eta \qquad \begin{vmatrix} 8-\lambda & -8 \\ 4 & -1-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 24 = 0$$
$$\eta' = 4\xi - \eta.$$

$\therefore \lambda = \frac{7 \pm i\sqrt{47}}{2}$ complex conjugate with +ve real part.

Hence the c.p. (2,4) is an unstable spiral point. (pg 210 case (b)).

3 (a). $x' = 3x - (x^2 + xy)$
$y' = -y + (xy - y^2)$

C.p. $x(3-x-y) = 0$, $y(-1+x-y) = 0 \Rightarrow x = 0$ or $3 - x - y = 0$, $y = 0$ or $-1 + x - y = 0$.

<u>Cases</u> If $x = 0$, then $y = 0$

$x = 0$, then $-1+x-y = -1-y = 0 \therefore y = -1$.

If $y = 0$, then $3-x-y = 3-x = 0 \quad \therefore x = 3$.

If $\begin{cases} -1+x-y = 0 \\ 3-x-y = 0 \end{cases}$

$2 \quad -2y = 0$ ie $y = 1$, $x = 2$.

Hence 4 c.p. $(0,0)$, $(0,-1)$, $(3,0)$ & $(2,1)$.

<u>Nature of c.p. $(0,0)$.</u>

$\varepsilon_1 = -(x^2+xy)$. Then $\frac{|\varepsilon_1|}{r} \le \frac{x^2+xy}{x^2} = x+y \to 0$ as $x \to 0$, $y \to 0$ (ie $r \to 0$).

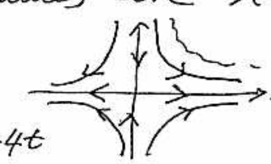
Similarly $\varepsilon_2 = o(r)$.

Hence the nature of $(0,0)$ is the same as that of the c.p. $(0,0)$ of the corr linear system

$$\begin{aligned} x' &= 3x \\ y' &= -y \end{aligned} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are $\lambda = 3, -1. \Rightarrow (0,0)$ unstable Saddle pt.

$x = x_0 e^{3t}$
$y = y_0 e^{-t}$



$y = \frac{c}{x^3}$ is the trajectory,

or $\frac{y}{x} = \frac{y_0}{x_0} e^{-4t}$

or $r^2 = x^2 + y^2 = x_0^2 e^{6t} + y_0^2 e^{-2t} \to \infty$ as $t \to \infty$. (unless $x_0 = 0$).

5/p220, 10(c).

$$x' = x - y^2$$
$$y' = y + xy$$

Consider

$$V = \frac{1}{2}(x^2+y^2) \geq 0$$

$$\therefore V' = x \cdot x' + y \cdot y'$$
$$= x(x-y^2) + y(y+xy)$$
$$= x^2+y^2 \geq 0$$

Hence the origin (0,0) is unstable, by Theorem 8.5.3 (pg 213)

Note: The nonlinear parts ($-y^2 + xy$) are $o(r)$.

The nature of c.p (0,0) can be deduced from:

$$x' = x$$
$$y' = y$$

Eigenvalues are $\lambda = 1, 1$.

See pg. 208 fig (c), with arrows reversed, ($y = cx$)

5/p220, 10(b)

$V = \frac{1}{2}(x^2+y^2)$. $V' = x(-y - x\sin^2 x) + y(x - y\sin^2 x)$
$= -\sin^2 x \cdot (x^2+y^2) \leq 0$

Hence the origin (0,0) is stable.