Winter '04

AMAT 411 Soln to #2

(1) (i) $y' = 1 + y^2,\ y(0) = 1$

$$\int_1^y \frac{dy}{1+y^2} = \int_0^x dx$$

$$\therefore \tan^{-1} y \Big]_1^y = x\Big]_0^x$$

$$\therefore \tan^{-1} y - \tan^{-1} 1 = x \qquad \therefore \tan^{-1} y = x + \frac{\pi}{4}$$

$$\therefore y = \tan\left(x + \frac{\pi}{4}\right), \qquad \left|x + \frac{\pi}{4}\right| < \frac{\pi}{2} \text{ or } -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

(ii) Note: If $ad \neq bc$, then use $y = vx$ (Homogeneous)

Here $ad = bc \quad \therefore \frac{a}{c} = \frac{b}{d} = k$, say.

Then

$$\frac{dy}{dx} = \frac{ckx + dky}{cx + dy} = k. \qquad \therefore y = kx + k_1$$
$$= \frac{a}{c}x + k_1$$

(2) $y' = y^{1/3} \sin 2x,\ y(0) = 0$

$$\int_0^y \frac{dy}{y^{1/3}} = \int_0^x \sin 2x\, dx$$

$$\therefore \frac{3}{2} y^{2/3} \Big]_0^y = -\frac{1}{2}\cos 2x\Big]_0^x$$

$$\therefore \frac{3}{2} y^{2/3} = -\frac{1}{2}(\cos 2x - 1)$$

$$\therefore y^{2/3} = \frac{1 - \cos 2x}{3}$$

$$\therefore y = \pm\left(\frac{1 - \cos 2x}{3}\right)^{3/2} \quad \text{are 2 solutions.}$$

The 3rd solution is $y(x) = 0$ for all x.

Since $f = y^{1/3}\sin 2x$, $\therefore f_y = \frac{\sin 2x}{3y^{2/3}}$ is not cts at $y=0$,

$\therefore$ The Ice & Uniqueness Theorem does not apply (ie we have more than one solutions).

3 (i) I.V.P. $y' = e^{-t^2} + y^2, \quad y(0) = 1$

Show: Solution $y(t)$ exists for $0 \le t \le \frac{\sqrt{2}}{1+(1+\sqrt{2})^2}$.

Since $f = e^{-t^2} + y^2$ & $f_y = 2y$ are cts for all (t, y)

$\therefore$ IVP has a unique solution for

$|t - 0| = |t| \le h = \min\left(a, \frac{b}{M}\right)$ or for

$0 \le t \le h$

Here $|t| \le a$, $|y-1| \le b$ or $y - 1 \le b \therefore y \le 1+b$

& $|f| = |e^{-t^2} + y^2| \le 1 + |y^2| \le 1 + (1+b)^2 = M$.

Let $g(b) = \frac{b}{M} = \frac{b}{1+(1+b)^2}$ To find max value of $g(b)$

$g'(b) = \frac{1+(1+b)^2 - b \cdot 2(1+b)}{1+(1+b)^2} = \frac{2-b^2}{1+(1+b)^2}$

$\therefore b = \sqrt{2}$

Since $g'(\sqrt{2}^-) = +ve$ & $g'(\sqrt{2}^+) = -ve$,

we have $b = \sqrt{2}$ max.

$\therefore$ Max. $g(b) = g(\sqrt{2}) = \frac{\sqrt{2}}{1+(1+\sqrt{2})^2}$.

Take $a = g(\sqrt{2})$.

$\therefore$ Soln exists for $0 \le t \le \frac{\sqrt{2}}{1+(1+\sqrt{2})^2}$

(ii). In this case, it is tedious to evaluate $g'(b)$.

IVP has a unique solution for

$0 \le t \le h = \min\left(a, \frac{b}{M}\right)$ say

Let $|t| \le a$, $|y - 0.4| < b = 0.6 \Rightarrow y < 0.4 + 0.6 = 1$.

Then $g(b) = \frac{b}{M} = \frac{0.6}{1+1} = 0.3$.

Take $a = 0.3$.

$\therefore$ Soln exists for $|t| \le h = \min(0.3, 0.3) = 0.3$.

(iii) Similar. $\left(g(b) = \frac{b}{1+\ln(1+b^2)} \to \infty, \text{ as } b \to \infty.\right)$

(4) Similar to prob. (1).
Note that if $|t| \le a$, then "a" can be chosen to be

$$a = \max\left(\frac{b}{M}\right) (= h).$$

(5) Gronwall's Lemma

If $c, f(x), g(x)$ are ≥ 0, &

$$f(x) \le c + \int_{x_0}^{x} f(s)g(s)\,ds, \quad \text{———} \quad (A)$$

then

$$f(x) \le c\, e^{\int_{x_0}^{x} g(s)\,ds}.$$

Proof: We have

$$\frac{f(x)}{c + \int_{x_0}^{x} f(s)g(s)\,ds} \le 1$$

$$\therefore \quad \frac{f(x).g(x)}{c + \int_{x_0}^{x} f(s)g(s)\,ds} \le g(x)$$

$$\therefore \quad \frac{d}{dx}\left(\ln\left(c + \int_{x_0}^{x} f(s)g(s)ds\right)\right) \le g(x)$$

Integrating from x_0 to x:

$$\int_{x_0}^{x} d\left(\ln\left(c + \int_{x_0}^{x} f(s)g(s)\,ds\right)\right) \le \int_{x_0}^{x} g(x)\,dx$$

$$\therefore \quad \ln\left(c + \int_{x_0}^{x} f(s)g(s)\,ds\right)\Big]_{x_0}^{x} \le \int_{x_0}^{x} g(x)\,dx$$

$$\therefore \ln\left(c + \int_{x_0}^{x} f(s)g(s)\,ds\right) - \ln c \le \int_{x_0}^{x} g(x)\,dx$$

$$\therefore \quad \ln \frac{c + \int_{x_0}^{x} f(s)g(s)\,ds}{c} \le \int_{x_0}^{x} g(x)\,dx$$

$$\therefore \quad c + \int_{x_0}^{x} f(s)g(s)\,ds \le c\,e^{\int_{x_0}^{x} g(x)\,dx}.$$

From (A):

$$f(x) \le c\,e^{\int_{x_0}^{x} g(s)\,ds} \quad \text{as required}$$

(6) $Y_0(x) = 1$,
$Y_n(x) = 1 + \int_0^x x^2 Y_{n-1}(x)\,dx = 1 + \frac{x^3}{3} + \left(\frac{x^3}{3}\right)^2 \frac{1}{2!} + \ldots$ Hence $Y(x) = \sum_0^\infty \left(\frac{x^3}{3}\right)^n \frac{1}{n!}$
$= e^{x^3/3}$.