1 (b) $(1-x^2)y'' - 2xy' + 2y = 0$

Since $y = x$ is a solution, we let $y = vx$.

Then $y' = v'x + v$,

$y'' = v''x + 2v'$, and so

$$(1-x^2)(v''x + 2v') - 2x(v'x + \cancel{v}) + 2\cancel{v}x = 0$$

$$x(1-x^2)v'' + 2v'(1 - x^2 - x^2) = 0$$

$$v'' + 2v'\frac{(1-2x^2)}{x(1-x^2)} = 0$$

Set $w = v'$. Then

$$\frac{dw}{w} = \frac{-2(1-2x^2)}{x(1-x^2)}\,dx. \qquad \text{partial fractions}$$

$$\ln\frac{w}{c} = \int \frac{-2}{x} - \frac{1}{1+x} + \frac{1}{1-x}\,dx.$$

$$= -2\ln x - \ln(1+x) - \ln(1-x)$$

i.e. $$\frac{w}{c} = \frac{1}{x^2(1+x)(1-x)}$$

$$\therefore v = c\int \frac{dx}{x^2(1+x)(1-x)} + c_1 \qquad \text{partial fractions}$$

$$= c\int \frac{1}{x^2} + \frac{1}{2}\frac{1}{1+x} + \frac{1}{2}\frac{1}{1-x}\,dx + c_1$$

$$= c\left[-\frac{1}{x} + \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)\right] + c_1$$

$$= c\left(-\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right) + c_1$$

Hence $$y = vx = c_2\left(-1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right) + c_1 x,$$

[illegible]

1 (d) $$y'' - 4xy' + 4x^2 y = xe^{x^2}.$$

Here $S - \frac{1}{2}R' - \frac{1}{4}R^2 = 4x^2 - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 2.$

Let $$u = e^{-\frac{1}{2}\int \frac{R}{\cancel{P}}dx} = e^{x^2}.$$

Set $$y = e^{x^2} v.$$

Then $$y' = e^{x^2}(v' + 2xv)$$

$$y'' = e^{x^2}(v'' + 4xv' + 2v + 4x^2 v)$$

$$\therefore\ e^{x^2}(v'' + \cancel{4xv'} + 2v \overset{+4x^2v}{\wedge}) - 4xe^{x^2}(\cancel{v'} + \cancel{2xv}) + 4x^2 e^{x^2} v = xe^{x^2}$$

or $$v'' + 2v = x.$$

The solution is.

$$v = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{x}{2}.$$

Hence

$$y = e^{x^2}\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{x}{2}\right).$$

2(a) $xy'' + (x-1)y' - y = 0$ ($x=0$ is reg. sing. pt.)

Try $y = \sum_{n=0} a_n x^{m+n}$, $a_0 \neq 0$.

We get

$$\sum_0 a_n(m+n)(m+n-1)x^{m+n-1} + \sum_0 a_n(m+n)x^{m+n}$$
$$-\sum_0 a_n(m+n)x^{m+n-1} - \sum_0 a_n x^{m+n} = 0$$

Equating to zero all coeff:

x^{m-1}: $a_0 m(m-2) = 0 \Rightarrow m = 0, 2$. (differ by integ

x^{m+n-1}: $a_n(m+n)(m+n-2) = -a_{n-1}(m+n-2)$, $n \geq 1$

$\therefore a_1(m+1)(m-1) = -a_0(m-1)$

$\therefore a_2(m+2)(m) = -a_1\cdot(m)$

$\therefore a_3(m+3)(m+1) = -a_2(m+1)$

Taking smaller root $m=0$, we see that

$a_1\cdot 1\cdot(-1) = -a_0\cdot(-1)$ $\therefore a_1 = -a_0$

$\therefore a_2\cdot 2\cdot 0 = -a_1\cdot 0 = 0$. Hence a_2 is arb.

$\therefore a_3 = \dfrac{-a_2\cdot\cancel{1}}{3\cdot 1} = -\dfrac{\cancel{1}}{3}a_2$ ~~etc~~

Hence the general solution is

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$$
$$= a_0(1+x) + a_2\left(x^2 - \frac{1}{3}x^3 + \cdots\right),$$ where a_0, a_2 are arb. const

$$= a_0(1-x) + \tilde{a}_2 \sum_2 \frac{(-x)^n}{n!}$$ where $\tilde{a}_2 = 2a_2$

$$= c_1(1-x) + \tilde{a}_2 \sum_0 \frac{(-x)^n}{n!}$$ where $c_1 = \cancel{a_0} + \tilde{a}_2$

$$= c_1(1-x) + \tilde{a}_2 e^{-x},$$ where $c_1, \tilde{a}_2$ are arb. const.

2(b) $x^2y'' - xy' + 8(x^2-1)y = 0$.

$x=0$ is a regular singular pt.

Let $y = \sum_0^\infty a_n x^{n+m} = a_0 x^m + a_1 x^{1+m} + \cdots \quad , (a_0 \neq 0)$. (1)

Then

$$\sum_0 a_n (n+m)(n+m-1)x^{n+m} - \sum_0 a_n (n+m) x^{n+m}$$
$$+ \sum_0 8 a_n x^{n+m+2} - \sum 8 a_n x^{n+m} = 0,$$

or

$$\sum_0 a_n[(n+m)(n+m-2) - 8]x^{n+m} + \sum_2 8 a_{n-2} x^{n+m} = 0.$$

Equate to zero, all coeff. of powers of x:

x^m: $\quad a_0[m(m-2)-8] = a_0(m+2)(m-4) = 0 \Rightarrow m = -2, 4$.

x^{1+m}: $\quad a_1[(1+m)(m-1)-8] = a_1(m^2-9) = 0 \Rightarrow a_1 = 0$

x^{n+m}: $\quad a_n(n+m+2)(n+m-4) = -8a_{n-2}, \quad n \geq 2$

We see that (i) when $m=4$, a_n are all defined

(ii) when $m=-2$, $a_6 = \frac{-8a_0}{0} \to \infty$.

So we replace a_0 by $b_0(m+2)$ in (1).

Then,

$$a_2 = \frac{-8a_0}{(m+4)(m-2)} = \frac{-8b_0(m+2)}{(m+4)(m-2)}$$

$$a_3 = 0, \quad a_5 = 0, \cdots$$

$$a_4 = \frac{-8a_2}{(m+6)(m)} = \frac{+8^2 b_0(m+2)}{(m+6)(m+4)m(m-2)}$$

$$a_6 = \frac{-8a_4}{(m+8)(m+2)} = \frac{-8^3 b_0 \cancel{(m+2)}}{(m+8)\cancel{(m+2)} \cdot (m+6)(m+4)m(m-2)} \quad \cancel{\frac{-8b_0}{m+8}} \text{ etc}$$

Thus the series

$$y = b_0 x^m \left[(m+2) - \frac{8(m+2)}{(m+4)(m-2)}x + \frac{8^2(m+2)x^4}{(m+6)(m+4)m(m-2)} - \cdots x^6\right].$$

is a solution when $m=-2$.

$$\therefore y_1 = y|_{m=-2} = b_0 x^{-2}\left[\frac{-8^3 x^6}{6 \cdot 4 \cdot 2 \cdot (-2)(-4)} + \cdots\right].$$

$$= \frac{-4}{3} b_0 x^4 \left(1 - \frac{x^2}{2} + \frac{x^4}{10} \cdots\right).$$

A 2nd soln is

$$y_2 = \frac{\partial y}{\partial m}\Big|_{m=-2} = y_1 \ln x \; \frac{b_0 3}{4} x^{-2}(1 + x^2 + x^4 \ldots)$$

2(d). Solutions are $y_1 = \frac{c_1}{x}$

$$y_2 = c_2\left(\frac{\ln x}{x} - \frac{1}{x^2} + \frac{1}{x^2}\sum_2^{\infty}\frac{t^n}{n!(n-1)}\right)$$

3. $y'' + y = f(x)$. $\quad y_c = c_1\cos x + c_2\sin x$.

t $y = c_1(x)\cos x + c_2(x)\sin x$.

en $$c_1' = \frac{\begin{vmatrix} 0 & \sin x \\ f(x) & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-f(x)\sin x}{1} \;;\; c_2' = f(x)\cos x.$$

$$\therefore c_1 = -\int f(x)\sin x\,dx + k_1 \;;\; c_2 = \int f(x)\cos x\,dx + k_2.$$

le can write:

$$c_1 = -\int_0^x f(t)\sin t\,dt + k_3 \;;\; c_2 = \int_0^x f(t)\cos t\,dt + k_4$$

Hence

$$y = -\int_0^x f(t)\cos x\sin t\,dt + k_3\cos x + \int_0^x f(t)\sin x\cos t\,dt + k_4\sin x$$

$$\therefore y_p = -\int_0^x f(t)\cos x\sin t + \int_0^x f(t)\sin x\cos t\,dt$$

$$= \int_0^x f(t)\left[-\cos x\sin t + \sin x\cos t\right]dt$$

$$= \int_0^x f(t)\sin(x-t)\,dt.$$