Winter 2004

AMAT 411

#3 (Solution)

2. $y' = Ay$ where

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 1 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -\lambda & 1 \\ 0 & -1 & \lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2+1) + \lambda^2+1 = (\lambda^2+1)(\lambda^2+1)$$

$\therefore \lambda = \pm i, \pm i$

(i) Take $\lambda = i$ ($\lambda = -i$ will give same result).

Find eigenvector v:

$$(A-iI)v = \begin{pmatrix} -i & 1 & 0 & 1 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

R_3: $-iv_3 + v_4 = 0$

R_4: $-v_3 - iv_4 = 0$ } same

We can choose $v_4 = 0$, $v_3 = 0$

Then $R_2 (= R_1)$: $-v_1 - iv_2 = 0$

Choose $v_2 = 1$, $v_1 = -i$

$$\therefore \quad v = \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence the solution: $e^{it}\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix} = (\cos t + i \sin t)\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\overset{\text{(split into real + imag. parts)}}{=} \begin{pmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} -\cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}.$

$\therefore$ We obtain 2 solutions:

$$y_1 = \begin{pmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix} = y_2$$

(ii) To find 2 more solutions, we find eigenvector w ($\neq v$):

$$(A-iI)^2 w = \begin{pmatrix} -i & 1 & 0 & 1 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix}\begin{pmatrix} -i & 1 & 0 & 1 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix} w$$

$$= \begin{pmatrix} -2 & -2i & -1 & -2i \\ 2i & -2 & 0 & -1 \\ 0 & 0 & -2 & -2i \\ 0 & 0 & 2i & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 (= R_4)$: $\quad -2w_3 - 2i\,w_4 = 0$

R_2 : $\quad 2i w_1 - 2w_2 - w_4 = 0$

Choose: $w_4 = 1$ (so that $w_4 \neq v_4$)

Then $w_3 = -i$

Choose $w_2 = 0$

Then $w_1 = \frac{-i}{2}$.

Hence the solution:

$$e^{it}\left(I + t(A-iI) + \cancel{\frac{t^2}{2}(A-\lambda I)^2} + \cdots\right) w$$

$$= e^{it}\left[\begin{pmatrix} -i/2 \\ 0 \\ -i \\ 1 \end{pmatrix} + t\begin{pmatrix} -i & 1 & 0 & 1 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix}\begin{pmatrix} -i/2 \\ 0 \\ -i \\ 1 \end{pmatrix}\right]$$

$$= e^{it}\left[\begin{pmatrix} -i/2 \\ 0 \\ -i \\ 1 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ i/2 \\ 0 \\ 0 \end{pmatrix}\right]$$

split up into real & imag. parts

$$= (\cos t + i\sin t)\left[\begin{pmatrix} -i/2 \\ 0 \\ -i \\ 1 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ i/2 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} \frac{t\cos t}{2} + \frac{\sin t}{2} \\ -\frac{t\sin t}{2} \\ \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} -\frac{\cos t}{2} + \frac{t\sin t}{2} \\ \frac{t\cos t}{2} \\ -\cos t \\ \sin t \end{pmatrix}$$

Hence 2 other solutions are:

$$Y_3 = \begin{pmatrix} \frac{t\cos t + \sin t}{2} \\ -\frac{t}{2}\sin t \\ \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{-\cos t + t\sin t}{2} \\ \frac{t}{2}\cos t \\ -\cos t \\ \sin t \end{pmatrix} = Y_4.$$

The general solution is.

$$Y(t) = c_1 Y_1 + c_2 Y_2 + c_3 Y_3 + c_4 Y_4 \quad \text{or}$$

$$= \begin{pmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

<u>Alternative method</u>

$$\begin{cases} y_1' = y_2 + y_4 \\ y_2' = -y_1 \\ y_3' = y_4 \\ y_4' = -y_3 \end{cases}$$

Last 2 eq: $\quad y_4'' = -y_3' = -y_4 \Rightarrow y_4'' + y_4 = 0 \quad \therefore y_4 = c_1\cos t + c_2\sin t$

etc.

3(a) $x' = -2x - 5y + 1$ Elimination Method or Other methods

$y'' = 2x + 5y + 2y'$

$\frac{d}{dt}$: $y''' = 2x' + 5y' + 2y'' = 2(-2x-5y+1) + 5y' + 2y''$

$= -4x - 10y + 2 + 5y' + 2y''$

$= -2(y'' - 5y - 2y') - 10y + 2 + 5y' + 2y''$

or

$y''' - 9y' = 2$ $m^3 - 9m = 0$ $m(m-3)(m+3) = 0$

Gen. Soln: $y = c_1 + c_2 e^{-3t} + c_3 e^{3t} - \frac{2}{9}t$

$x = \frac{1}{2}(y'' - 2y' - 5y)$

$= \frac{1}{2}\left[9c_2 e^{-3t} + 9c_3 e^{3t} - 2\left(-3c_2 e^{-3t} + 3c_3 e^{3t} - \frac{2}{9}\right) - 5\left(c_1 + c_2 e^{-3t} + c_3 e^{3t} - \frac{2}{9}t\right)\right]$

$= -\frac{5c_1}{2} + 5c_2 e^{-3t} - c_3 e^{3t} + \frac{2}{9} + \frac{5}{9}t$

Applying initial cond:

(b) $\left.\begin{aligned} -\frac{5}{2}c_1 + 5c_2 - c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ -3c_2 + 3c_3 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$

Soln is $x = \frac{2}{9} + \frac{5t}{9}$, $y = -\frac{2}{9}t$

(4) (a) Elimination Method

$$y'' = -x' + 1$$
$$= -(4x+3y-2-4t)+1$$
$$= +4(y'-t) - 3y + 3 + 4t$$

or

$$y'' - 4y' + 3y = 3$$

or

$$y = y_c + y_p$$
$$= c_1 e^{t} + c_2 e^{3t} + 1$$
$$\therefore \quad x = t - y' = t - c_1 e^{t} - 3c_2 e^{3t}$$

(b) Matrix Method

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2-4t \\ t \end{pmatrix}$$

Find fundamental matrix solution $X(t) = \begin{pmatrix} -e^{t} & -3e^{3t} \\ e^{t} & e^{3t} \end{pmatrix}$

Then use

$$\begin{pmatrix} x \\ y \end{pmatrix} = X(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + X(t)\int_0^t X^{-1}(t)\begin{pmatrix} -2-4t \\ t \end{pmatrix} dt \quad \text{etc.}$$

4. <u>Method A</u> (Elimination)

$$\begin{cases} x' = 4x + 3y - 2 - 4t \\ y' = -x + t \end{cases}$$

$$(D-4)x - 3y = -2 - 4t \quad \text{—— (1)}$$
$$x + Dy = t \quad \text{—— (2)}$$

(1) − (D−4)(2)

$$[-3-(D-4)D]\,y = -2-4t-(D-4)t = -2-4t-(1-4t) = -3$$

$$+(D^2-4D+3)\,y = +3$$

$$m^2-4m+3 = (m-1)(m-3) = 0 \Rightarrow m = +1, +3.$$

$$\therefore y = y_c + y_p$$
$$= \underline{c_1e^{+t} + c_2e^{+3t} + 1}$$

From (2):

$$x = t - y'$$
$$= t - (+c_1e^{+t} + 3c_2e^{+3t}) = \underline{-c_1e^{t} - 3c_2e^{3t} + t}$$

or <u>Use fl. matrix.</u>

$$\begin{pmatrix} x \\ y \end{pmatrix} = Y(t)c + Y(t)\int_0^t Y^{-1}(s)\begin{pmatrix} -2-4s \\ s \end{pmatrix} ds.$$

AMAT 411
#3 Solution

5.(a) If $x = uv$, $y = wv + z$, where (u, w) is a solution of

$$x' = ax + by,$$
$$y' = cx + dy.$$

then we get

$$x' = u'v + uv' = auv + b(wv + z)$$
$$y' = w'v + wv' + z' = cuv + d(wv + z)$$

or

$$uv' = auv + b(wv + z) - u'v$$
$$= (au + bw - u')v + bz.$$

$$\therefore v' = \frac{bz}{u}.$$

and

$$z' = cuv + d(wv + z) - w'v - wv'$$
$$= (cu + dw - w')v + dz - w\left(\frac{bz}{u}\right)$$
$$= \left(d - \frac{wb}{u}\right)z, \quad \text{as required.}$$

(b) $(t, 1)$ is a solution of

$$x' = x + (1-t)y$$
$$y' = \frac{x}{t} - y.$$

Then if we set $x = tv$, $y = v + z$, we get

$$\begin{cases} v' = \dfrac{(1-t)}{t} z. \\ z' = \left(-1 - \dfrac{(1-t)}{t}\right) z \end{cases}$$

Solving the second eq., $\dfrac{dz}{z} = \dfrac{-1}{t}\, dt$.

$$\therefore z = \frac{c}{t}.$$

$$\therefore v = c\int \frac{1-t}{t^2}\, dt = c\int \frac{1}{t^2} - \frac{1}{t}\, dt$$

$$= c_1 + c\left(-\frac{1}{t} - \ln t\right).$$

Hence
$$\begin{cases} x = t\left(c_1 + c\left(-\frac{1}{t} - \ln t\right)\right) = c_1 t - c\,(1 + t\ln t), \\ y = c_1 + c\left(-\frac{1}{t} - \ln t\right) + \frac{c}{t} = c_1 - c\ln t. \end{cases}$$