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THE FUTURE OF MATHEMATICS VIEWED FROM 1800

The discovery of the calculus was followed by a century of brilliant exploitation led by men like Euler and Lagrange. But towards the end of that century it became clear that several problems of physics and pure mathematics were resisting all attacks. In addition the lack of any clear and agreed foundation for the calculus led to paradoxes and obscurities which might be ignored but could not be resolved.

In a much quoted letter to d'Alembert in 1781, Lagrange wrote

> It seems to me that the mine is already almost too deep, and unless we discover new seams we shall sooner or later have to abandon it. Today Physics and Chemistry offer more brilliant and more easily exploited riches; and it seems that the taste of the century has turned entirely in that direction. It is not impossible that the mathematical positions in the Academies will one day become what the University chairs in Arabic are now.
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> (p. 386, Vol. 13 of Lagrange's *Collected Works*).

These sentiments were echoed by Delambre in his summary for the Academy of Sciences of the state of the mathematical sciences ('Rapport Historique sur les Progrès des Sciences Mathématiques Depuis 1789'). Speaking of the calculus he described the general view of such discoveries. 'Prepared by the work of many centuries, when they are sufficiently ripe to be plucked by genius, their first effect is to excite admiration and their second to impose on following generations immense labours whose glory will never equal their difficulty.'

> It would be difficult (he continues) and perhaps rash to try and analyse the chances which the future offers for the progress of mathematics; in almost all its branches we are halted by insurmountable difficulties and only improvements of detail seem to remain for us. All movements which cannot be reduced to small oscillations about a stable point subject to simple laws, all solutions which we are not given initially to a first approximation, all these things seem beyond our grasp...

After remarking that the fundamental principles of hydrodynamics still eluded mathematicians, he goes on: – 'All these difficulties seem to show that the power of our methods is almost exhausted, just as the power of ordinary algebra was in the time of Leibnitz and Newton and that we need some new method to enable us to study transcendental functions and the equations which contain them.'

Delambre was mathematical secretary to the Academy of Sciences and his report appeared in 1810. Three years previously the prefect of Isère had submitted to the Academy a memoir on the conduction of heat which was received without enthusiasm. Looking back, we now see this memoir as heralding the surge of new mathematical methods and results which were to mark the new century.

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WHO WAS FOURIER? I

(The contents of the next two chapters are based on the excellent biography by Herivel *Joseph Fourier The Man and the Physicist*, Oxford 1975.)

Joseph Fourier was born in 1768 in Auxerre, the ninth child of a master tailor. Although the death of his father left him an orphan at the age of ten, his intelligence gained him a free place at the local Benedictine school. At the end of a brilliant school career he applied to enter the artillery only to be informed that such a profession was only open to those of noble blood and was closed to him 'even if he were a second Newton'.

Fourier began to prepare to enter the Benedictine teaching order but, whatever his plans may have been, the course of his life was violently altered by the outbreak of the French Revolution. 'As the natural ideas of equality developed, it was possible to conceive the sublime hope of establishing among us a free government exempt from kings and priests and to free from this double yoke the long usurped soil of Europe. I readily became enamoured of this cause, in my opinion the greatest and the most beautiful which any nation has undertaken.'

However, the Revolution was soon threatened by problems of its own making. The collapse of royal authority and the effects of revolutionary zeal and political misjudgement created administrative chaos whilst at the same time driving France into war against most of Europe. A succession of military defeats aggravated by treachery, economic problems and internal revolution put the Republic in danger and brought Robespierre's party to power. In the next two years France was reorganised for war and central power reasserted. Both at a national and a local level, educated men were needed to fill the administrative vacuum caused by the revolution. Fourier, who was now teaching at his old school, became first an active member of the revolutionary committee of Auxerre and then its president.

The situation of the new Republic called for ruthless measures which the government, conscious of its own revolutionary virtues, was well prepared to take. Treachery was fought by a political terror in which opponents both to the left and right were executed and, as the definition of treachery was extended, it became clear that no one was safe. Fourier himself was arrested, released and then rearrested. A deputation from Auxerre which, with considerable courage, went to Paris to plead his case, was told – 'Yes, he speaks well, but we no longer have any need of musical patriots.' Only the fall of Robespierre saved Fourier's head.

However Fourier's release did not mark the end of his troubles. As coup d'état followed coup d'état, and the revolution swung erratically to the right he would remain a marked man. No one had been executed in Auxerre but Fourier had been an agent of the terror there. His arrest was on a charge of Hébertism and the Hébertists were to the left of Robespierre. The word 'terrorist' then, like 'Trotskyist' now, denoted a defeated yet feared opponent.

Luckily an opportunity to leave Auxerre now presented itself. A new college (the Ecole Normale) was being set up in Paris to help train teachers and Fourier was nominated as a pupil by a neighbouring district. Fourier could now study under men like Lagrange, Monge and Laplace and escape his terrorist past. Fourier's talents were soon noted, but the college was not successful and its closure was followed by further problems for Fourier.

'We shudder when we think that the pupils of the Ecole Normale were chosen under the reign of Robespierre and his protégés. It is only too true that Balme and Fourier, pupils of the department of Yonne have long professed the atrocious principles and infernal maxims of the tyrants. Nevertheless they prepare to become teachers of our children. Is it not to vomit their poison in the bosom of innocence (From an address to the National Convention, quoted by Herivel).' Fourier was again arrested, released, rearrested and finally, following yet another political swing, released to become a teacher at the new Ecole Polytechnique.

Here Fourier remained for three years. That his talent was recognised is shown by the fact that he succeeded Lagrange in the Chair of Analysis and Mechanics. This quiet interlude was ended by a government order to join the invasion of Egypt. Ostensibly intended to liberate Egypt from the Turks and to threaten the British position in India, the expedition may have been seen by the government as a way of keeping a troublesome general as far away as possible and by the general (Napoleon) as the first step toward becoming Emperor of the East. Fourier was one of a group of scientists and intellectuals intended to form part of the immense cultural benefits that France was to bestow on Egypt.

Unfortunately the effects of stunning French victories on land were negated by Nelson's destruction of the French invasion fleet. Learning of serious military and domestic problems in France, Napoleon deserted his army and rushed home to 'save France'. A rapid and successful coup d'état enabled him to read the dispatches in which General Kléber (his successor in Egypt) explained exactly what he thought of this behaviour.

Both before and after Napoleon's departure, Fourier occupied several important administrative and political posts in Egypt. When the French expedition finally surrendered in 1801 and Fourier was repatriated, Napoleon offered him the post of Prefect of the Department of the Isère centred round Grenoble. (France had been divided into 83 Departments and each Prefect governed his Department on behalf of the central government.) Although he could have continued as Professor at the Polytechnique, Fourier accepted the offer. Herivel suggests that Egypt had given him a taste for administration and that he hoped to rise higher. Herivel also suggests that Fourier's close association with Kléber after Napoleon's departure accounts for the fact that these hopes were not fulfilled.

Fourier seems to have been a popular and efficient Prefect. His greatest achievement during his 14 years of office was by reconciling the conflicting interests of some forty communities to enable the swamps of Bourgoin to be drained. The draining of twenty thousand acres of swamps resulted in major economic and health benefits and was achieved during a period more noted for grandiose paper plans than for concrete achievements. Fourier's other administrative memorial was a new road across the Alps (now Route N91).

Apart from his prefectorial duties Fourier helped organise the *Description of Egypt*. This work written by the intellectuals attached to the Egyptian expedition did much to inspire European interest in Egypt and was thus one of the two permanent results of that expedition. (The other was the discovery of the Rosetta Stone, a trilingual inscription which was to provide the key to the deciphering of hieroglyphics.) Fourier's main contribution was the general introduction – a survey of Egyptian history up to modern times. (An Egyptologist with whom I discussed this described the introduction as a masterpiece and a turning point in the subject. He was surprised to hear that Fourier also had a reputation as a mathematician.) On a personal level he encouraged Champollion, a linguistic infant prodigy, to take up Egyptology and used his position as prefect to preserve his protegé from conscription. It was Champollion who eventually deciphered the Rosetta Stone.

Napoleon's domestic policy carried to its natural conclusion a political compromise which had been developing since the fall of Robespierre. The old powers of the church and the nobility were to be reconciled with the new power of the bourgeoisie and an administration loyal to the state alone. Henceforward men were to be equal before the law but the state renounced any attempt to achieve economic equality, or indeed any form of economic control. Fourier who had risen through his own talent but who 'could give lessons in theology to bishops and politeness to the pre-1790 parliamentarians' both symbolised and in his position as Prefect aided the new compromise.

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WHO WAS FOURIER? II

'Yesterday was my 21st birthday, at that age Newton and Pascal had [already] acquired many claims to immortality.' Fourteen years after he wrote the postscript above, Fourier was prefect of Isère but had still no claim to the immortality he had craved as a young man. The work on the zeros of algebraic polynomials which he had pursued since his time at Auxerre still gains him a footnote in algebra textbooks – but a footnote is not immortality. His lecturing while at the Ecole Polytechnique had been much praised – but a good lecture is the most ephemeral of triumphs.

However, in 1804 he took up the subject of the propagation of heat. The field of Newtonian mechanics had already been worked over by several masters but the physics of heat, light, electricity and magnetism had still to be brought under the rule of mathematics. The choice of one of these fields thus requires no explanation. The particular choice of heat may have been connected with an obsessional need for warmth which Fourier acquired in Egypt.

In three remarkable years Fourier found the fundamental equations for heat conduction, developed new methods to solve them, applied these methods in a wide variety of cases and produced experimental evidence to support his solutions. At the end of 1807 he had submitted a memoir containing this work to the Academy. A commission consisting of Lagrange, Laplace, Monge and Lacroix was set up to examine it.

Instead of the enthusiastic acceptance that Fourier may have hoped for, he ran into two major criticisms. On the mathematical side neither Laplace nor Lagrange could accept the validity of his use of Fourier series. For example Laplace could not believe that $\cos x$ could possibly be expressed using a sine series. On the physical side Laplace and his pupils had already attacked the problem of heat conduction from a different angle. Unable to accept that Fourier's approach was superior, Laplace, Biot and Poisson attacked his derivation of the equations of conduction as lacking in rigour.

In 1811 the Academy gave Fourier a second chance to promote his theory by offering its grand prize in mathematics for work on the theory of heat conduction.

Fourier resubmitted his earlier essay together with some further results. (The most important addition was the introduction of the Fourier transform and its use, in the manner of Chapter 55, in studying the cooling of infinite solids.) However, although Fourier gained the prize, the accompanying report made it clear that Lagrange and Laplace had not withdrawn their objections.

The attitude of Lagrange and Laplace has been severely criticised by writers of modern texts on mathematical methods as (to quote one of them) 'typical in its demonstration of the difference between pure and applied mathematicians'. Since Laplace was the greatest mathematical physicist of the age and the author of the *Traité de Mécanique Céleste*, whilst Lagrange (who admittedly often worked in pure mathematics) was the author of the equally influential *Mécanique Analytique* this seems a curious judgement.

In fact Laplace and, particularly, Lagrange, did not simply doubt the rigour but also the truth of Fourier's assertions concerning expansions in trigonometric series. They may also have doubted the utility of such expansions. Why should Fourier's expansion in trigonometric series be superior to an expansion in continued fractions, or infinite products, or formal power series or any of the hundred and one pretty tricks of the eighteenth century mathematicians? With hindsight we can see (or at least persuade ourselves that we can see) why Fourier series are much better adapted than Taylor series to a whole class of physical problems. Laplace and Lagrange could not see into the future and their doubts are surely more a tribute to the originality of Fourier's methods than a reproach to mathematicians who Fourier greatly respected (and, in Lagrange's case, admired).

Soon new political worries were added to Fourier's academic ones (the Academy had published neither his first nor his second memoir leaving his results open to plagiarism or rediscovery). Napoleon had continued the aggressive foreign policy of the revolution with results depicted in vast battle pieces by a generation of French painters and, on a smaller scale, by Goya. So long as French forces were victorious the cost of the policy would be borne by the defeated powers and not by France but from 1812 onwards the tide of war turned. In 1814 enemy troops entered Paris, Napoleon was forced to abdicate and Louis XVIII installed as a constitutional monarch.

The new king understood, even if some of his followers did not, that the Napoleonic domestic compromise would have to form the basis of his rule. Nor could all of France's experienced administrators be replaced by the few men who had stayed loyal to him in exile. Fourier remained in Grenoble as prefect of the Isère.

However the last act of the Napoleonic adventure was still to be played. The new government was short of money and made the mistake of trying to economise on the pensions of Napoleon's old armies, and, worse, the pension of Napoleon himself. Tired of his role as a ruler of the little kingdom of Elba, Napoleon returned to France and began a march towards Paris. Troops sent out to arrest him went over to him without firing a shot. Grenoble was the first large town on Napoleon's line of march. Fourier's attempts to organise resistance proved fruitless and he and the commanding general left Grenoble by one gate as Napoleon entered by another.

Whilst at Grenoble, Fourier had acted as a loyal and indeed zealous servant of the King. But the continuing collapse of the Royalist position seems to have persuaded him to throw in his lot with Napoleon and five days later we find him as Napoleon's prefect of the Rhône.

Twenty years before when Fourier was president of the revolutionary committee of Auxerre, the republic had lacked everything from experienced soldiers to gunpowder; but hope and enthusiasm had saved France. Now there was material and organisation but only Napoleon's promise of a new regime that was to be at once revolutionary, liberal and imperial. Passive and active resistance increased and Fourier himself was dismissed for failure to comply with orders. Napoleon was defeated at Waterloo and the King returned.

Fourier was now without a job and in deep disfavour with the new regime. The pension to which his years of public service would normally have entitled him was refused on technical grounds. An internal minute concerning his case sums up the government's attitude. 'The prefect of Grenoble at the time of the arrival of Bonaparte should not be surprised not to have a pension.'

The low point of Fourier's fortunes did not last long. The prefect of Paris was an old pupil and friend and had gained sufficient credit with the new government to be able to appoint him as Director of the Statistical Bureau of the Seine. (There are demographers who only know of Fourier as an important figure in the development of French governmental statistics.) In 1816 he was elected to the Academy of Sciences but the King refused to confirm his election. In 1817 the Academy tried again and this time his election was allowed. The physical and mathematical results of his theory of heat began to gain general acceptance. In 1822 his prize essay was at last published and he was elected permanent mathematical secretary of the Academy.

Fourier was now able to enter upon the role of grand old man encouraging younger talent (Liouville, Sturm, Dirichlet, Navier) and attaining that curious summit of French distinction, election to the Académie Française. He did not completely stop mathematical work and some of his later work concerns problems in what is now called linear programming. (Darboux, the nineteenth-century editor of his collected work clearly wondered why such a great mind should be interested in such out of the way puzzles.)

'There was at the Academy of Sciences', wrote Victor Hugo, seeking a contrast with the growing fame of the socialist Charles Fourier, 'a celebrated Fourier whom posterity has forgotten'. But posterity has not forgotten Joseph Fourier the Egyptologist, mathematician, physicist and public servant.