Solutions by series

Example Solve $y'' + y = 0$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots$

so $y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$

Substitute into the ODE to get

$$(a_0 + 2a_2) + (a_1 + 6a_3)x + (a_2 + 12a_4)x^2 + (a_3 + 20a_5)x^3 + \cdots = 0$$

Since this is to be an identity, each coefficient must vanish

i.e. $a_0 + 2a_2 = 0, \quad a_1 + 6a_3 = 0, \quad a_2 + 12a_4 = 0, \quad a_3 + 20a_5 = 0 \quad \cdots$

so $a_2 = -\frac{a_0}{2}, \quad a_3 = \frac{-a_1}{6} = -\frac{a_1}{3!} \quad a_4 = \frac{-a_2}{12} = \frac{a_0}{4!} \quad a_5 = \frac{-a_3}{20} = \frac{+a_1}{5!}$

hence $y = a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots$

or $y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$

Recall from calculus that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

so even if you never heard of $\sin x$ or $\cos x$ you would be led to them by the ODE.

Remark: Can prove the addition law directly from the differential equation.

Exercise: Solve $y' = y$ by series

Example Solve $y'' + 2xy' - y = 0$ Subject to $y(0) = 0$, $y'(0) = 1$

Solution $y = \sum a_j x^j$ (think $\sum_{j=0}^{\infty}$)

$y' = \sum j a_j x^{j-1}$

$y'' = \sum j(j-1)\, a_j x^{j-2}$

so $\sum j(j-1)\, a_j x^{j-2} + 2x \sum j a_j x^{j-1} - \sum a_j x^j = 0$

or, to get all the terms x^j together, replace j by $j+2$ in first sum. Get

$$\sum (j+2)(j+1)\, a_{j+2} x^j + \sum 2j a_j x^j - \sum a_j x^j = 0$$

[[Recall $a \sum u_j = \sum a u_j$, and $\sum_{k=a}^{b} u_k = \sum_{k=0}^{b-a} u_{k+a}$]]

or $\sum \left[(j+2)(j+1)\, a_{j+2} + (2j-1)\, a_j \right] x^j = 0$

so $(j+2)(j+1)\, a_{j+2} + (2j-1)\, a_j = 0$

$$a_{j+2} = \frac{-(2j-1)}{(j+1)(j+2)}\, a_j$$

Put $j = 0, 1, 2, \ldots$ obtain

$a_2 = \dfrac{a_0}{2\cdot 1}$, $a_3 = \dfrac{-a_1}{3\cdot 2}$ $a_4 = \dfrac{-3a_2}{4\cdot 3} = -\dfrac{3}{4!}\, a_0$

$a_5 = \dfrac{-5a_3}{5\cdot 4} = \dfrac{5}{5!}\, a_1$ $a_6 = \dfrac{-7}{6\cdot 5}\, a_4 = \dfrac{3\cdot 7}{6!}\, a_0$ $a_7 = \dfrac{-9a_5}{7\cdot 6} = -\dfrac{5\cdot 9}{7!}\, a_1$ et

Hence

$$y = a_0\left(1 + \frac{x^2}{2!} - \frac{3x^4}{4!} + \frac{3\cdot 7}{6!}x^6 - \frac{3\cdot 7\cdot 11}{8!}x^8 + \cdots\right)$$

$$+ a_1\left(x - \frac{x^3}{3!} + \frac{5x^5}{5!} - \frac{5\cdot 9x^7}{7!} + \frac{5\cdot 9\cdot 13}{9!}x^9 - \cdots\right)$$

From $y(0)=0$, $y'(0)=1$ get $a_0=0$ $a_1=1$. Therefore, desired solution is

$$y = x - \frac{x^3}{3!} + \frac{5x^5}{5!} - \frac{5\cdot 9}{7!}x^7 + \frac{5\cdot 9\cdot 13}{9!}x^9 - \cdots$$

This series is not related in any obvious manner to any elementary functions which we know. But, can still plot a graph, give it a name, if it was important the discoverer could go down in history etc.

Q1: How do we know the series is the solution? Bit of a vicious circle since we do not know we have the solution until we ~~kno~~ have $a_0, a_1, a_2, \dots$

Try theoretical approach in special case:

(*) $\quad p(x)\,y'' + q(x)\,y' + r(x)\,y = 0 \quad$ assume $p\,q\,r$ polynomials or analytic

Definition: A point x for which $p(x) = 0$ is called a singular point of the differential equation. Any other point is called a regular point or ordinary point.

Example $\quad (x^2+1)\,y'' - 2y' + xy = 0 \quad$ has singular points $\pm i$

Theorem: (1) The general solution of (*) can be found by power series about $x=a$:

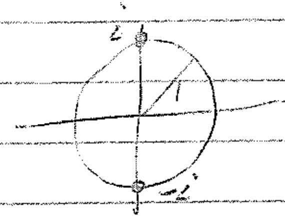
$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$

if a is an ordinary point.

(2) The series solutions in (1) converge for all x such that $|x-a| < R$, where R is the distance from a to nearest singularity in the complex plane. The series diverges for $|x-a| > R$.

example: Consider series $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$

Radius of convergence is 1
Function is real analytic on all of $\mathbb{R}$

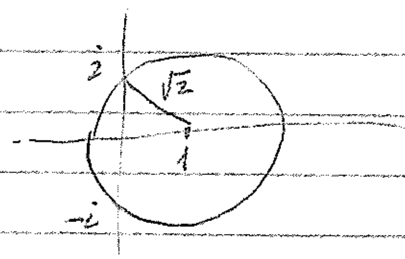


Example $(x^2+1)y'' - 2y' + 5xy = 0 \quad a = 0, 1$

Singularity at $x = \pm i$. If $a = 0$, Series $a_0 + a_1 x + a_2 x^2 + \cdots$

converges for $|x| < 1$. It may or may not converge at $x = \pm 1$

If $a = 1$



$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots$$

Then series solution converges for $1-\sqrt{2} < x < 1+\sqrt{2}$

Example Solve $xy'' - y = 0 \quad y(2) = 0 \quad y'(2) = 3$

Change variable $v = x-2 \qquad (v+2)\dfrac{d^2y}{dv^2} - y = 0 \qquad y = 0 \quad \dfrac{dy}{dv} = 3 \quad v = 0$

Assume $y = a_0 + a_1 v + a_2 v^2 + \cdots = \sum a_k v^k$

$$(v+2)\frac{d^2y}{dv^2} - y = (v+2)\sum k(k-1)a_k v^{k-2} - \sum a_k v^k$$

$$= \sum k(k-1)a_k v^{k-1} + \sum 2k(k-1)a_k v^{k-2} - \sum a_k v^k$$

$$= \sum \left[k(k+1)a_{k+1} + 2(k+1)(k+2)a_{k+2} - a_k \right] v^k = 0$$

so conclude

$$a_{k+2} = \frac{a_k - k(k+1)a_{k+1}}{2(k+1)(k+2)}$$

Now $y = a_0$, $\frac{dy}{dv} = a_1$ at $v = 0$, so $a_0 = 0$ $a_1 = 3$

Put $k = 0, 1, 2, \cdots$ into recurrence, find

$$a_2 = 0 \quad a_3 = \frac{1}{4} \quad a_4 = -\frac{1}{16}, \quad a_5 = \frac{1}{40} \quad a_6 = \frac{-3}{320}, \ldots$$

So $$y = 3v + \frac{v^3}{4} - \frac{v^4}{16} + \frac{v^5}{40} - \frac{3v^6}{320} + \cdots$$

$$= 3(x-2) + \frac{1}{4}(x-2)^3 - \frac{1}{16}(x-2)^4 + \frac{1}{40}(x-2)^5 - \frac{3}{320}(x-2)^6 + \cdots$$

and converges on $0 < x < 4$

Note: This time got a three term recurrence. Does not really cause difficulty, but tends to obscure what is going on in general.

Picard's Method of Iteration

(Mainly used to prove existence and uniqueness theorems)

Example Solve $y' = x + y + 1$

Assume $y = c$ when $x = 0$, integrate to get

$$y = c + \int_0^x (x + y + 1)\, dx$$

Of course, can not do integral. But assume $y_1 = c$ is a first approximation to y. Put this in integral

$$y_2 = c + \int_0^x (x + y_1 + 1)\,dx = c + \int_0^x (x + c + 1)\,dx = c + \frac{x^2}{2} + cx + x$$

Iterate: $$y_3 = c + \int_0^x (x + y_2 + 1)\,dx = c + (c+1)x + \frac{(c+2)}{2!}x^2 + \frac{x^3}{3!}$$

also $$y_4 = c + (c+1)x + (c+2)\frac{x^2}{2!} + \frac{(c+2)x^3}{3!} + \frac{x^4}{4!}$$

in general

$$y_n = c + (c+1)x + \frac{(c+2)x^2}{2!} + \frac{(c+2)x^3}{3!} + \cdots + \frac{(c+2)x^{n-1}}{(n-1)!} + \frac{x^n}{n!}$$

In the limit $$\lim_{n\to\infty} y_n = c + (c+1)x + \frac{(c+2)x^2}{2!} + \frac{(c+2)x^3}{3!} + \cdots$$

Can check this with the closed form solution

Exercise: Solve $y' = xy$ $\quad y(0) = 5$ by series.
Sum the series in closed form

Same for $xy'' + y' = 0$ $\quad y(1) = 2$ $\quad y'(1) = 3$

Method of Frobenius

Consider $\quad p(x)\, y'' + q(x)\, y' + r(x)\, y = 0 \qquad (1)$

Can one find a power series like solution about a if a is a singular point? For example, suppose initial conditions were given at $x=a$.

Started with Euler differential equations

Saw example on assignment 2, in general

$$\left(a_n x^n D^n + a_{n-1} x^{n-1} D^{n-1} + \cdots + a_1 x D + a_n\right) y = F(x)$$

can be transformed by $x = e^z$ to constant coefficient case

Example:

① $\quad 2x^2y'' + 2xy' + y = 0 \qquad y = c_1 \frac{1}{x} + c_2 \frac{1}{\sqrt{x}}$

② $\quad x^2y'' \cancel{\pm} 2xy' + 2y = 0 \qquad y = c_1 x^2 + c_2 x$

③ $\quad x^2y'' - xy' + y = 0 \qquad y = c_1 x + c_2 x \ln x$

Suppose you assumed sol'n $y = a_0 + a_1 x + a_2 x^2 + \dots = \sum_i a_n x^n$

get general sol'n for ②, 1 solution for ③ and none for ①

Frobenius' IDEA: Try more general form of solution

$$y = x^c\left(a_0 + a_1 x + a_2 x^2 + \dots\right) = \sum_i a_j x^{j+c} \qquad *$$

Not too hard to show any Euler equation of order 2 or higher has at least 1 solution of type (*).

Call these solutions Frobenius-type solutions.

Q: What conditions on $p(x), q(x), r(x)$ where ~~#~~ $x=0$ is a singular point imply that the differential equation has a solution of Frobenius type (*)?

Don't know what Frobenius was thinking, but maybe something like:

Compare (1) with $k_2 x^2 y'' + k_1 xy' + k_0 y = 0$ $\quad k_i$ constants

~~&~~ better, divide: $\quad y'' + \frac{q(x)}{p(x)} y' + \frac{r(x)}{p(x)} y = 0$

$$y'' + \frac{k_1}{k_2 x} y' + \frac{k_0}{k_2 x^2} y = 0$$

Of course, if $\frac{k_1}{k_2 x} = \frac{q(x)}{p(x)}$ $\quad \frac{k_0}{k_2 x^2} = \frac{r(x)}{p(x)}$ $\quad$ (1) is an Euler equation

BUT if not, but still have $\lim_{x \to 0} \frac{x(q(x))}{p(x)} = \text{constant}$

$\lim_{x \to 0} \frac{x^2 r(x)}{p(x)} = \text{constant}$

Then (1) is not an Euler equation but ought to be nearly an Euler equation if x is close enough to zero. So not unreasonable to expect (1) to have a solution of Frobenius type.

Example $\quad (2x^2+5x^3)\,y'' + (2x-x^2)\,y' + (1+x)\,y = 0$

if x is small, expect that a good approximation to a solution will be a solution of

$$2x^2 y'' + 2x y' + y = 0$$

This is ① which has $\frac{c_1}{x} + \frac{c_2}{\sqrt{x}}$ as general solution

Definition $\quad$ Let $x=a$ be a singular point of the ODE (1).
Then if

$$\lim_{x\to a} \frac{(x-a)\,q(x)}{p(x)} \quad \text{and} \quad \lim_{x\to a} \frac{(x-a)^2\, r(x)}{p(x)}$$

both exist, call $x=a$ a <u>regular</u> singular point. Otherwise it is an <u>irregular</u> singular point.

Example $\quad x^3(1-x)\,y'' + (3x+2)\,y' + x^4 y = 0$

$p(x) = x^3(1-x) \quad q(x) = (3x+2) \quad r(x) = x^4 \quad$ Singular points at $0, 1$.

$\lim_{x\to 0} \frac{(x)(3x+2)}{x^3(1-x)}$ does not exist $\quad \lim_{x\to 0} \frac{(x^2)(x^4)}{x^3(1-x)} = 0$

$\lim_{x\to 1} \frac{(x-1)(3x+2)}{x^3(1-x)} = -5 \quad \lim_{x\to 1} \frac{(x-1)^2(3x+2)}{x^3(1-x)} = 0$

so $\quad 0$ – irregular singular point $\quad 1$ – regular singular point

THEOREM Let $x=a$ be a regular singular point of

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0$$

with p, q, r analytic functions about $x=a$.

Then the differential equation has at least one solution of Frobenius' type of form

$$y = (x-a)^c\left[a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots\right] = \sum a_k (x-a)^{k+}$$

and the series $\sum a_k (x-a)^k$ converges in $|x-a| < R$ where R is the distance from a to the nearest singularity.

S There is no claim that all solutions are of Frobenius type.

Examples $4x\,y'' + 2y' + y = 0$ $p = 4x$ $q = 2$ $r = 1$

$x=0$ singular point

$$\lim_{x\to 0} \frac{x\,q(x)}{p(x)} = \lim_{x\to 0} \frac{x\cdot 2}{4\cdot x} = \frac{1}{2} \qquad \lim_{x\to 0} \frac{x^2\,r(x)}{p(x)} = \lim_{x\to 0} \frac{x^2\cdot 1}{4x} = ($$

So 0 is a regular singular point. So get at least one solution of Frobenius type

$$y = \sum a_k x^{k+c}$$

$y' = \sum_1 (k+c)x^{k+c-1} \qquad y'' = \sum_1 (k+c)(k+c-1)a_k x^{k+c-2}$

Substitute into diff eqn, get

$$4xy'' + 2y' + y = \sum_1 [4(k+c)(k+c-1)]a_k x^{k+c-2} + \sum_1 2(k+c)a_k x^{k+c-1} + \sum a_k x^{k+c}$$

relabel

$$= \sum_1 4(k+c+1)(k+c)a_{k+1} x^{k+c} + \sum_1 2(k+c+1)a_{k+1} x^{k+c} + \sum_1 a_k x^{k+c}$$

$$= \sum_1 \{[4(k+c+1)(k+c) + 2(k+c+1)]a_{k+1} + a_k\} x^{k+c}$$

$$= \sum_1 \{(2k+2c+2)(2k+2c+1)a_{k+1} + a_k\} x^{k+c} = 0$$

This implies

$$(2k+2c+2)(2k+2c+1)a_{k+1} + a_k = 0 \qquad (\dagger)$$

Since $a_k = 0$ for $k < 0$, first value of k for which we get any information is $k = -1$. When $k = -1$, $(\dagger)$ is

$$(2c)(2c-1)a_0 = 0, \quad \text{or since } a_0 \neq 0$$

$$(2c)(2c-1) = 0 \quad \text{from which } c = 0, \tfrac{1}{2}$$

This equation for determining c is called the <u>indicial equation</u>; the values of c are the <u>indicial roots</u>

So we have 2 cases to consider (next time)

Case 1: $c=0$ Then $(2k+2c+2)(2k+2c+1)a_{k+1}+a_k=0$

is $(2k+2)(2k+1)a_{k+1}+a_k=0$ or $a_{k+1} = -\dfrac{a_k}{(2k+1)(2k+2)}$

put $k=0,1,2,\dots$ find

$$a_1 = -\frac{a_0}{2!} \qquad a_2 = -\frac{a_1}{3\cdot 4} = \frac{a_0}{4!} \qquad a_3 = -\frac{a_2}{5\cdot 6} = -\frac{a_0}{6!} \quad \text{etc}$$

Hence

$$y = \sum a_k x^k = a_0\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots\right)$$

Case 2 $c=\frac{1}{2}$ Then $(2k+2c+2)(2k+2c+1)a_{k+1}+a_k=0$

is $(2k+3)(2k+2)a_{k+1}+a_k=0$ $a_{k+1} = -\dfrac{a_k}{(2k+2)(2k+3)}$

put $k=0,1,2,$ etc

$$a_1 = -\frac{a_0}{3!} \qquad a_2 = -\frac{a_1}{4\cdot 5} = \frac{a_0}{5!} \qquad a_3 = -\frac{a_2}{6\cdot 7} = -\frac{a_0}{7!} \quad \text{etc}$$

Hence $$y = \sum a_k x^{k+1/2} = a_0\left(x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \dots\right)$$

The general solution is

$$y = A\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + B\left(x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \dots\right)$$

$$= A\cos\sqrt{x} + B\sin\sqrt{x} \, .$$

Note: Series converge for all x since $x=0$ is only singular point.

Example Find Frobenius type solution of

$$x^2y'' + xy' + (x^2-1)y = 0 \qquad p(x) = x^2 \quad q(x) = x \quad r(x) = x^2-1$$

$x=0$ is a singular point

$$\lim_{x\to 0} \frac{x\,q(x)}{p(x)} = \lim_{x\to 0} \frac{x\cdot x}{x^2} = 1 \qquad \lim_{x\to 0} \frac{x^2 r(x)}{p(x)} = \lim_{x\to 0} \frac{x^2(x^2-1)}{x^2} = -1$$

So $x=0$ is a regular singular point, so try $y = \sum a_k x^{k+c}$

Substitute, get

$$x^2 \sum (k+c)(k+c-1)\,a_k x^{k+c-2} + x\sum (k+c)\,a_k x^{k+c-1} + (x^2-1)\sum a_k x^{k+c} = 0$$

rewrite as

$$\sum \left[(k+c)(k+c-1)\,a_k + (k+c)\,a_k + a_{k-2} - a_k \right] x^{k+c} = 0$$

simplify to

$$(k+c+1)(k+c-1)\,a_k + a_{k-2} = 0$$

Put $k=0$, get $(c+1)(c-1)\,a_0 = 0$, yields indicial equation

$$(c+1)(c-1) = 0 \quad \text{so} \quad c = 1, -1.$$

Case 1: $c=-1$ $\quad k(k-2)\,a_k + a_{k-2} = 0$ or

$$a_k = \frac{-a_{k-2}}{k(k-2)}$$

Put $k=1$ get $a_1 = 0$, put $k=2$ not meaningful since assumed $a_0 \neq 0$. So no solution in this case.

Case 2: $c = 1$ $\quad (k+2)k\, a_k + a_{k-2} = 0$ or $a_k = -\dfrac{a_{k-2}}{k(k+2)}$

put $k = 1, 2, 3,$ get

$$a_1 = 0 \quad a_2 = -\frac{a_0}{2\cdot 4} \quad a_3 = 0 \quad a_4 = -\frac{a_2}{4\cdot 6} = \frac{a_0}{2\cdot 4^2\cdot 6}, \quad a_5 = 0, \ldots$$

$$a_6 = -\frac{a_4}{6\cdot 8} = -\frac{a_0}{2\cdot 4^2\cdot 6^2\cdot 8}$$

so $$y = \sum a_k x^{k+1} = a_0\left(x - \frac{x^3}{2\cdot 4} + \frac{x^5}{2\cdot 4^2\cdot 6} - \frac{x^7}{2\cdot 4^2\cdot 6^2\cdot 8} + \cdots\right)$$

and this series converges for all x since $x=0$ is only singular point.

In general, know that if one solution, say $Y_1(x)$ of a linear ODE is known, can find a second by reduction of order, i.e. $y = v\,Y_1(x)$

Example: Solve $x^2 y'' + xy' + (x^2-1)\,y = 0$

We already have 1 Frobenius type solution, so let $y = v\,Y_1$

so $y' = vY_1' + v'Y_1$, $\quad y'' = vY_1'' + 2v'Y_1' + v''Y_1$

so differential equation becomes

$$x^2\left(vY_1'' + 2v'Y_1' + v''Y_1\right) + x\left(vY_1' + v'Y_1\right) + (x^2-1)\,vY_1 = 0$$

or $$x^2 v'' Y_1 + (2x^2 Y_1' + xY_1)\,v' + \left(x^2 Y_1'' + xY_1' + (x^2-1)Y_1\right]v = 0$$

Since Y_1 satisfies the ODE this reduces to

$$x^2 v'' Y_1 + (2x^2 Y_1' + x Y_1) v' = 0$$

This is 1^{st} order in v' let $u = v'$, $x^2 Y_1 u' + (2x^2 Y_1' + x Y_1) u = 0$

or $$\frac{u'}{u} + \frac{2Y_1'}{Y_1} + \frac{1}{x} = 0$$

or $$\ln u + 2\ln Y_1 + \ln x = \ln A \quad \text{or} \quad \ln(u Y_1^2 x) = \ln A$$

or $$u Y_1^2 x = A \quad \text{or} \quad v' = \frac{A}{x Y_1^2} \quad \text{so} \quad v = A\int \frac{dx}{x Y_1^2} + B$$

Since $y = v Y_1$, obtain $$y = A Y_1 \int \frac{dx}{x Y_1^2} + B Y_1$$

as the general solution. Put $a_0 = 1$ in for Y_1: $Y_1 = x - \frac{x^3}{2\cdot 4} + \frac{x^5}{2\cdot 4^2\cdot 6} - \ldots$

$$\therefore \frac{1}{Y_1} = \frac{1}{x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \ldots\right)} = \frac{1}{x}\left(1 + \frac{x^2}{8} + \alpha_1 x^4 + \ldots\right)$$

α_1 a constant which don't need explicitly for present purposes.

$$\therefore \frac{1}{x Y_1^2} = \frac{1}{x^3}\left(1 + \frac{x^2}{4} + \alpha_2 x^4 + \ldots\right)$$

so $$\int \frac{1}{x Y_1^2}\,dx = -\frac{1}{2x^2} + \frac{1}{4}\ln x + \frac{\alpha_2 x^2}{2} + \ldots$$

So $Y_2 = \frac{1}{4} Y_1 \ln x + F$, F a Frobenius type series.

the Presence of $\ln x$ explains why we did not get 2 Frobenius-type solutions.

Example (Only need consider 1 indicial root) $xy'' + 2y' - xy = 0$

$p(x) = x \quad q(x) = 2 \quad r(x) = -x \qquad x = 0$ a singularity

$$\lim_{x\to 0} \frac{xq(x)}{p(x)} = \lim_{x\to 0} \frac{x\cdot 2}{x} = 2 \qquad \lim_{x\to 0} \frac{x^2 r(x)}{p(x)} = \lim_{x\to 0} \frac{x^2\cdot(-x)}{x} = 0$$

$\therefore$ $x=0$ is a regular singular point. Try $y = \sum a_k x^{k+c}$

Get $\sum (k+c)(k+c-1) a_k x^{k+c-1} + 2\sum (k+c) a_k x^{k+c-1} - \sum a_k x^{k+c+1} = 0$

or, $(k+c+1)(k+c+2)\, a_{k+1} - a_{k-1} = 0$

Put $k=-1$ in, yields $k(k+1)\, a_0 = 0$ since $a_0 \neq 0$, get $c = -1, 0$ as indicial roots.

Case 1: $c = -1$ $\qquad k(k+1)\, a_{k+1} - a_{k-1} = 0 \quad (1)$ or $a_{k+1} = \dfrac{a_{k-1}}{k(k+1)}$

if $k=0$ get a_1 is arbitrary, since (1) is $0 \cdot a_1 - 0 = 0$

Now subst $k = 1, 2, 3, \dots$

$$a_2 = \frac{a_0}{1\cdot 2} = \frac{a_0}{2!} \qquad a_3 = \frac{a_1}{2\cdot 3} = \frac{a_1}{3!} \qquad a_4 = \frac{a_2}{3\cdot 4} = \frac{a_0}{4!} \qquad a_5 = \frac{a_3}{4\cdot 5} = \frac{a_1}{5!}$$

$$\therefore \quad y = \sum a_k x^{k+c} = \sum a_k x^{k-1}$$

$$= a_0\left(x^{-1} + \frac{x}{2!} + \frac{x^3}{4!} + \frac{x^5}{5!} + \dots\right) + a_1\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right)$$

$$= a_0 \frac{\cosh x}{x} + a_1 \frac{\sinh x}{x}$$

Case 2 $c=0$. No need since already have general solution from case 1.

(This example shows why we try the smaller indicial root first)

Example (repeated roots) Find a Frobenius type solution about $x=1$ for

$$x(x-1)\,y'' + xy' + y = 0$$

$$\lim_{x\to 1} \frac{(x-1)\,q(x)}{p(x)} = \lim_{x\to 1} \frac{(x-1)(x)}{x(x-1)} = 1 \qquad \lim_{x\to 1} \frac{(x-1)^2\, r(x)}{p(x)} = \lim_{x\to 1} \frac{(x-1)^2(1)}{x(x-1)} = 0$$

$\therefore$ $x=1$ is a regular singular point. So try $y = \sum a_k (x-1)^{k+c}$

Change variable by $v = x-1$ get

$$v(v+1)\frac{d^2y}{dv^2} + (v+1)\frac{dy}{dv} + y = 0 \qquad y = \sum a_k v^{k+c}$$

Substitute, get

$$\sum \left\{ \left[(k+c)^2+1\right] a_k + (k+c+1)^2 a_{k+1} \right\} v^{k+c} = 0$$

so $$\left[(k+c)^2+1\right] a_k + (k+c+1)^2 a_{k+1} = 0$$

put $k=-1$, $c^2 a_0 = 0$ ie $c=0$ is a repeated root

Sub $c=0$ $$(k^2+1)\,a_k + (k+1)^2 a_{k+1} = 0$$ or

$$a_{k+1} = \frac{-(k^2+1)}{(k+1)^2}\, a_k$$

$a_1 = -a_0 \quad a_2 = -\frac{(1^2+1)}{2^2} a_1 = \frac{(1^2+1)}{2^2} a_0 \quad a_3 = -\frac{(2^2+1)}{3^2} a_2 = -\frac{(1^2+1)(2^2+1)}{2^2 3^2} a_0 \ \ldots$

$V = x-1$

So $$y = a_0\left[1 - (x-1) + \frac{(1^2+1)}{2^2}(x-1)^2 - \frac{(1^2+1)(2^2+1)}{2^2 3^2}(x-1)^3 + \cdots\right]$$

only get 1 solution in this case.

Typically, see:

1. If indicial roots differ by a non integer, ie. not $0, \pm 1, \pm 2, \cdots$ always get general solution
2. If indicial roots differ by an integer, 2 possibilities ($\neq 0$ differe
   (a) No solution from smaller root. But always get one solution from larger root
   (b) Get the general solution from the smaller root
3. If indicial root is repeated (ie. differ by integer $= 0$) only 1 solution is obtained.

Bessel's Differential Equation

(Table of useful series?)

Consider $y'' + y = 0$. Even if you never heard of sin or cos, you would be led to them from this ODE. You can even prove the addition formula directly from the ODE (Exercise)

So IDEA is that given an ODE, may find new functions as solutions. This takes on a new significance if the ODE is from an important applied problem. This is why many important mathematical discoveries in the 1800s were made by scientists and engineers.

Example The equation $x^2 y'' + xy' + (x^2 - n^2) y = 0$

is Bessel's equation of order n. n can be any value, (even complex!) but usually is an integer.

Note: $x=0$ is a regular singular point, so solution of form $y = \sum_k a_k x^{k+c}$

Substitute, grind to get $\sum_k \left[\{(k+c)^2 - n^2\} a_k + a_{k-2} \right] x^{k+c} = 0$

so $\{(k+c)^2 - n^2\} a_k + a_{k-2} = 0$

Put $k=0$, get $c^2 - n^2 = 0$, so indicial roots are $\pm n$

Case 1: $c = n > 0$ $\quad a_k = \dfrac{-a_{k-2}}{k(2n+k)}$

put $k=1$, $a_1 = 0$, and so $a_3 = a_5 = a_7 = a_9 = \cdots$ are all 0 as well

Then

$$a_2 = \frac{-a_0}{2(2n+2)} \qquad a_4 = \frac{a_0}{2\cdot 4\,(2n+2)(2n+4)} \qquad a_6 = \frac{-a_0}{2\cdot 4\cdot 6\,(2n+2)(2n+4)(2n+6)} \quad \text{etc}$$

So $$y = a_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4\,(2n+2)(2n+4)} + \cdots \right]$$

$$= a_0 x^n \left[1 - \frac{(x/2)^2}{1!\,(n+1)} + \frac{(x/2)^4}{2!\,(n+1)(n+2)} - \frac{(x/2)^6}{3!\,(n+1)(n+2)(n+3)} + \cdots \right]$$

Trick: let $a_0 = \dfrac{1}{2^n n!}$

so $$y = \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{(x/2)^2}{1!(n+1)!} + \frac{(x/2)^4}{2!(n+2)!} - \frac{(x/2)^6}{3!(n+3)!} + \cdots \right]$$

This is a solution for $n = 0, 1, 2, \ldots$ with convention $0! = 1$

For n to be any positive number, define generalization of factorial

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du \qquad x > 0$$

then $\Gamma(x+1) = x\,\Gamma(x) \qquad \Gamma(1) = 1$

then $\Gamma(2) = 1\cdot\Gamma(1) = 1! \quad \Gamma(3) = 2\,\Gamma(2) = 2!$ and $\Gamma(n+1) = n!$ if n is a positive integer. Can also show $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

so $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \dfrac{\sqrt{\pi}}{2}$ etc

Give names to things so we can talk about them, so

Definition The Bessel function of order n (of the first kind) is

$$J_n(x) = \frac{(x/2)^n}{\Gamma(n+1)} - \frac{(x/2)^{n+2}}{1!\,\Gamma(n+2)} + \frac{(x/2)^{n+4}}{2!\,\Gamma(n+3)} - \frac{(x/2)^{n+6}}{3!\,\Gamma(n+4)} + \cdots$$

Case 2 $c = -n$, $n \geq 0$ Don't have to do everything again, just replace n by $-n$, get

$$* \quad J_{-n}(x) = \frac{(x/2)^{-n}}{\Gamma(-n+1)} - \frac{(x/2)^{-n+2}}{1!\,\Gamma(-n+2)} + \frac{(x/2)^{-n+4}}{2!\,\Gamma(-n+3)} - \cdots$$

How to interpret Γ for negative numbers: Integral doesn't converge but can use functional relation $x\Gamma(x) = \Gamma(x+1)$

$$\text{ie.} \quad \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi} \qquad \Gamma\left(-\tfrac{3}{2}\right) = \frac{4}{3}\sqrt{\pi} \quad \text{etc}$$

Functional relation yields $0 \cdot \Gamma(0) = 1$ or $0 = \frac{1}{\Gamma(0)}$ so say $\Gamma(0)$ is infinite. Consistent to say $\frac{1}{\Gamma(-1)} = 0$, $\frac{1}{\Gamma(-2)} = 0$ etc

Thus $J_{-n}(x)$ becomes defined for all negative numbers.

If $n > 0$ but not an integer, $*$ is not bounded at $x = 0$ while J_n is. Therefore general solution is

$$y = c_1 J_n(x) + c_2 J_{-n}(x) \qquad n \neq 0, 1, 2, 3, \cdots$$

If n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$ $n = 0, 1, 2, 3, \ldots$

exercise: Check for $n = 3$.

Use reduction of order, get second solution, so form general sol'n

$$y = C_1 J_n(x) + C_2 J_n(x) \int \frac{dx}{x[J_n(x)]^2}$$

If n is not an integer, $Y_n(x) = \dfrac{\cos n\pi \, J_n(x) - J_{-n}(x)}{\sin n\pi}$ is also a solution, independent of $J_n(x)$

If n is an integer, get indeterminate form $\frac{0}{0}$, take limit with L'Hopital's Rule.

Call this second solution (for any n) the Bessel function of the second kind of order n.

So can say $y = A J_n(x) + B Y_n(x)$

is general solution for any n.

These functions look like damped sine waves, and indeed are asymptotic to them

They satisfy many identities and relations, such as

1. $\dfrac{d}{dx}\left[x^n J_n(x) \right] = x^n J_{n-1}(x)$

2. $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$

Generating function: $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)\, t^n$

Exercise: Show $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x$

Problem: To show that the function

$$F_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

really is the Bessel function $J_n(x)$ for n an integer ≥ 0.

Solution: The technique is to write $F_n(x)$ as a power series about $x=0$ and see that it agrees with known power series for $J_n(x)$

$$F_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

write $F_n(x) = \sum_0^{\infty} \frac{1}{k!} F_n^{(k)}(0) \cdot x^k$

Differentiate under the integral:

$$F_n'(x) = \frac{1}{2\pi}\int_0^{2\pi} -\sin(n\theta - x\sin\theta)\cdot(-\sin\theta)\, d\theta$$

$$F_n''(x) = \frac{1}{2\pi}\int_0^{2\pi} -\cos(n\theta - x\sin\theta)\,(-\sin\theta)^2\, d\theta$$

$\vdots$

$$F_n^{(k)}(x) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} (-1)^{k/2}\cos(n\theta - x\sin\theta)\,(-\sin\theta)^k\, d\theta & k \text{ even} \\ \frac{1}{2\pi}\int_0^{2\pi} (-1)^{\frac{k+1}{2}}\sin(n\theta - x\sin\theta)\,(-\sin\theta)^k\, d\theta & k \text{ odd} \end{cases}$$

when $x=0$ these reduce to

$$F^{(k)}(0) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} (-1)^{k/2}\cos n\theta \sin^k\theta\, d\theta & k \text{ even} \\ \frac{1}{2\pi}\int_0^{2\pi} (-1)(-1)^{\frac{k+1}{2}} \sin n\theta \sin^k\theta\, d\theta & k \text{ odd} \end{cases}$$

k even

set $a_{nk} = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \sin^k\theta \; d\theta$

set $z = e^{i\theta} = \cos\theta + i\sin\theta \qquad z^{-1} = \cos\theta - i\sin\theta$

so $z^n = e^{in\theta} = \cos n\theta + i\sin n\theta \Rightarrow \cos n\theta = \frac{z^n + z^{-n}}{2}$

$\sin\theta = \frac{z - z^{-1}}{2i}$

$a_{nk} = \frac{1}{2\pi}\int \frac{z^n + z^{-n}}{2} \cdot \frac{(z - z^{-1})^k}{(2i)^k} \frac{dz}{iz}$

Coefficient of 1 in $(z^n + z^{-n})(z - z^{-1})^k$

is $z^n \, {}_kC_a \, z^a z^{-(k-a)} \Rightarrow n + a - k + a = 0 \Rightarrow a = \frac{k-n}{2}$

$+ \; z^{-n} \, {}_kC_b \, z^b z^{-(k-b)} \Rightarrow -n + b - k + b = 0 \Rightarrow b = \frac{k+n}{2}$

$(-1)^{k-a} \, {}_kC_a = (-1)^{\frac{n+k}{2}} \frac{k!}{\left(\frac{k-n}{2}\right)!\left(\frac{k+n}{2}\right)!} \qquad (-1)^{k-b} \, {}_kC_b = (-1)^{k-b} \frac{k!}{\left(\frac{k+n}{2}\right)!\left(\frac{k-n}{2}\right)!}$

$(-1)^{k-a} + (-1)^{k-b} = (-1)^{\frac{k-n}{2}} + (-1)^{\frac{k+n}{2}} = (-1)^{k/2} \cdot \left[(-1)^{-n/2} + (-1)^{n/2}\right] = (-1)^{k/2} \cdot 2 \cdot (-1,$

$= 2 \cdot (-1)^{\frac{n+k}{2}}$

residue is $\quad 2 \cdot (-1)^{\frac{n+k}{2}} \cdot \frac{1}{2} \cdot \frac{1}{2^k (-1)^{k/2}} \cdot \frac{k!}{\left(\frac{k-n}{2}\right)! \left(\frac{k+n}{2}\right)!}$

$$= (-1)^{n/2} \cdot \frac{1}{2^k} \cdot \frac{k!}{\left(\frac{k-n}{2}\right)! \left(\frac{k+n}{2}\right)!}$$

<u>Note</u> $\quad \int_0^{2\pi} \cos n\theta \sin^k\theta \, d\theta = \int_{-\pi}^{\pi} \cos n\theta \sin^k \theta \, d\theta = 0 \quad$ if k is odd

If k is even, n must be even as well.

~~If k is odd~~ By residue theorem, conclude

$$F^{(k)}(0) = (-1)^{\frac{n+k}{2}} \cdot \frac{1}{2^k} \cdot \frac{k!}{\left(\frac{k-n}{2}\right)! \left(\frac{k+n}{2}\right)!} \qquad \text{for } k \text{ even}$$

If k is odd set $\quad b_{nk} = \frac{1}{2\pi} \int_0^{2\pi} \sin n\theta \sin^k\theta \, d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2i} \left(z^n - z^{-n}\right) \left(\frac{z - z^{-1}}{(2i)^k}\right)^k \cdot \frac{dz}{iz}$$

Want the coefficient of 1 in $\left(z^n - z^{-n}\right)\left(z - z^{-1}\right)^k$

Computation is same as last time except we subtract instead of add:

Now $(-1)^{k-a} - (-1)^{k-b} = (-1)^{\frac{k-n}{2}} - (-1)^{\frac{k+n}{2}} = -2 \cdot (-1)^{\frac{n+k}{2}}$

so residue is $-2 \cdot (-1)^{\frac{n+k}{2}} \cdot \frac{1}{2} \cdot (-1)^{\frac{k+1}{2}} \cdot \frac{1}{2^k} \cdot \frac{k!}{\left(\frac{k-n}{2}\right)!\left(\frac{k+n}{2}\right)!}$

$$= (-1)^{\frac{n+1}{2}} \cdot \frac{1}{2^k} \cdot \frac{k!}{\left(\frac{k-n}{2}\right)!\left(\frac{k+n}{2}\right)!}$$

Note:

In this case, both n & k are odd

The case <u>n even</u>

$$F_n(x) = \sum_{k \geq n}^{\infty} (-1)^{\frac{n+k}{2}} \cdot \frac{1}{k!} \frac{1}{2^k} \frac{k!}{\left(\frac{k-n}{2}\right)!\left(\frac{k+n}{2}\right)!} x^k \qquad n, k \text{ even}$$

$$= \sum_{k \geq n}^{\infty} (-1)^{\frac{n+k}{2}} \frac{1}{\left(\frac{k-n}{2}\right)!\left(\frac{k+n}{2}\right)!} \left(\frac{x}{2}\right)^k$$

put $\ell = k-n$ $\qquad \ell$ even

$$= \left(\frac{x}{2}\right)^n \sum_{0}^{\infty} (-1)^{\ell/2} \frac{1}{(\ell/2)!\left(\frac{\ell+2n}{2}\right)!} \left(\frac{x}{2}\right)^\ell$$

put $\ell = 2j$

$$= \left(\frac{x}{2}\right)^n \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(j+n)!} \left(\frac{x}{2}\right)^{2j} = J_n(x) \quad \text{for } n \text{ even}$$

The case n odd

$$F_n(x) = \sum_{k \geq n}^{\infty} (-1) \cdot (-1)^{\frac{k+1}{2}} \cdot (-1)^{\frac{n+1}{2}} \cdot \frac{1}{2^k} \cdot \frac{k!}{\left(\frac{k-n}{2}\right)! \left(\frac{k+n}{2}\right)!} \frac{x^k}{k!}$$

$$= \sum_{k \geq n}^{\infty} (-1)(-1)^{\frac{n+k+2}{2}} \frac{1}{\left(\frac{k-n}{2}\right)! \left(\frac{k+n}{2}\right)!} \left(\frac{x}{2}\right)^k$$

put $l = k-n$ so l is even

$$= \sum_{0}^{\infty} (-1)^{\frac{l+2n+2}{2}} \frac{1}{(l/2)! \left(\frac{l+2n}{2}\right)!} \left(\frac{x}{2}\right)^{l+n}$$

$$= \left(\frac{x}{2}\right)^n \sum_{0}^{\infty} (-1)^{l/2} \frac{1}{(l/2)! \left(\frac{l+2n}{2}\right)!} \left(\frac{x}{2}\right)^l$$

put $2j = l$

$$= \left(\frac{x}{2}\right)^n \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(j+n)!} \left(\frac{x}{2}\right)^{2j}$$

$$= J_n(x) \quad \text{for } n \text{ odd}.$$