1. Let F be the filter defined by

$$y_t = x_t - x_{t-1},$$

and let G be the filter defined by

$$y_t = x_t + y_{t-1}.$$

Consider the two filters H_1, which takes the output of F and uses it as the input of G, and H_2, which takes the output of G and uses it as the input of F.

   (a) Explain why F is a discrete differentiator, and G is a discrete integrator.

   (b) Determine the output of H_1 and H_2 if we give the unit impulse δ_t as input. Note: this is of course easy to do by hand, but you may wish to write a Python script that computes this, just to help you get up to speed writing scripts.

   (c) Explain why this should be viewed as a discrete version of the fundamental theorem of calculus.

2. Suppose that a given filter has the frequency response

$$H(e^{i\omega}) = \frac{1 - \frac{1}{2}e^{-i\omega} + e^{-i3\omega}}{1 + \frac{1}{2}e^{-i\omega} + \frac{3}{4}e^{-i3\omega}}.$$

   (a) Give a defining equation of the filter (i.e. the recurrence relation for the output y_t in terms of the input x_t and the previous outputs y_{t-k}.)

   (b) Plot the zeroes and poles of the transfer function $\mathscr{H}(z)$ in the complex plane.

   (c) Explain why the filter is or is not stable.

   (d) Using a computer, plot the frequency response of the filter. The abscissa should have units of frequency, in fractions of the sampling rate, and the ordinate should have the units of decibels, just like the figures in the text such as figure 1.2 on page 103.

3. This exercise explores the discrete analogue of the well known result from the theory of ordinary differential equations that an n^{th} order ordinary differential equation with constant coefficients has solutions of the form $e^{a_k t}$, and the numbers a_k are roots of the auxiliary polynomial.

   (a) Consider the special case of the famous Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, which is the recurrence with initial values $y_1 = 1$, $y_2 = 1$, and difference equation

   $$y_t = y_{t-2} + y_{t-1}.$$

   Assume that $y_t = r^t$ for some number r, and substitute into the difference equation. What quadratic polynomial does r satisfy? Let the roots of this polynomial be r_1 and r_2.

   (b) Show that the recurrence y_t has solutions of the form

   $$y_t = \sum_{m=1}^{2} c_m r_m^t$$

   where the c_ms are arbitrary constants and the r_ms are the 2 roots of the polynomial. Find values of the constants c_1 and c_2 so that y_t satisfies the initial conditions $y_1 = 1$, $y_2 = 1$.

4. Let y_t satisfy

   $$y_t - 5y_{t-1} - 6y_{t-2} = 0$$

   and $y_0 = 0$ and $y_1 = 1$. Find y_t in two ways. First, write a script that will produce the output, and second, solve it analytically.

5. (Extra for experts.) By once again examining the case of constant coefficient differential equations, deduce what happens to the solution of such a difference equation in the case of a double root. Likely you will get stuck on doing the general case all at once, so make up an example that you can solve easily to see what is going on. For example, one that has two roots z_1, z_2 and try something like $z_1 = 2$ and $z_2 = 2 + \epsilon$ and letting $\epsilon \to 0$ or solving a difference equation that has auxiliary roots $z_1 = z_2 = 2$ directly.