Name ______________________________

1. Suppose that two signals

$$\begin{aligned} x_1 &= A\cos 100\pi t \\ x_2 &= A\cos 120\pi t \end{aligned}$$

are combined. What is the resulting beat frequency?

Solution: Recall the trigonometric identity

$$\cos(a+b)+\cos(a-b) = 2\cos a\cos b$$

Setting $a+b = 120\pi t$, $a-b = 100\pi t$, we get $a = 110\pi t$, $b = 10\pi t$, so the combined signal is

$$x_1 + x_2 = 2A\cos 110\pi t\cos 10\pi t = 2A\cos 2\pi 55t\cos 2\pi 5t$$

This says that what you hear is a sine wave of frequency 55 Hz whose volume is modulated at a frequency of 5 Hz (recall that a sine wave of frequency f Hz is given by $x(t) = \sin 2\pi ft$ where the time t is in seconds.)

2. What ratio of amplitudes is represented by one bel?

Solution: One bel is ten decibels, and when referring to measurements of amplitude we consider the ratio of the squares of the two amplitudes A_2 and A_1. This is because power is proportional to the square of amplitude.

$$1\,\text{bel} = 10\,\text{decibels} = 10\log_{10}\left(\frac{A_2^2}{A_1^2}\right) = 20\log_{10}\left(\frac{A_2}{A_1}\right)$$

This implies that $A_2/A_1 = 10^{1/2} = \sqrt{10} = 3.16227766\ldots$

3. Suppose that you blow across one end of a straw, leaving the other end open. Then suppose that you block the other end with a finger. Predict what will happen to the pitch. Verify experimentally.

Solution: Physics tells us that the fundamental vibration mode has a node at a closed end and an antinode at an open end. Thus, one would expect that an open tube has a fundamental with a standing wave whose wavelength is one half of the length of the tube (two antinodes, one node) and a half closed tube (for pessimists, or half open if you are an optimist) would have a fundamental whose wavelength is one quarter the length of the tube (one node, one antinode). Hence one would expect the pitch to drop by a factor of two (since the wavelength has doubled) or one octave.

In my experiment, I used a straw 14cm long which gives a musical note of D. Indeed, closing the end off drops the pitch an octave. As a side note, this explains why a flute is closed off at one end, since otherwise, it would have to be twice as long to give a note of the same pitch. This basic fact that you have to double the length of a tube to halve the frequency explains why musical instruments have to undergo some strange contortions to be able to play those low notes. For example, the tube on a contrabassoon is 14 ft long. You can also hunt around on youtube for examples of contrabass saxophones and flutes just to see how strange this is.

4. Suppose that you pluck a guitar string, then put your finger in the centre of the string, damping the motion of that spot. What do you think will happen to the spectrum of the sound? Verify experimentally.

   Solution: This is a little tricky to actually do. It helps to pluck the string near the bridge of the guitar, so that you excite many harmonics in the vibration. Momentarily damping the string at the twelfth fret kills the fundamental mode, and just leaves all the higher harmonics, of which the second is the loudest. This means that you will hear the same note, but an octave higher. Notice also that this technique may be used to great effect at the fifth and seventh frets as well, and doing this produces what are called natural harmonics. There are also artificial harmonics, which may be played at any position on the instrument, and require much more skill as a player to be able to do. Such notes are required when one plays Bach's chorale from Cantata No. 147 (Jesu, joy of man's desiring) on bass quitar. They are also played by harp players, and as a treat one should go and hear the harpist play in the Calgary philharmonic orchestra sometime.

5. Suppose we have a signal $x(t) = \cos \omega t$. Letting $y = 2x^2 - 1$, it follows that $y(t) = \cos 2\omega t$. Hence, $y(t)$ is the second harmonic.

(a) Work out the next case $\cos 3\omega t$ in terms of $\cos \omega t$.

(b) Prove the general fact that the result for every integral harmonic is always a polynomial. (These polynomials are called Chebyshev polynomials.)

(c) Generating harmonics is a form of waveshaping, see the notes in the text on page 58

Solution: We set

$$e^{i3\theta} = \cos 3\theta + i \sin 3\theta = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$$

Expanding, and equating the real and imaginary parts and remembering that $\sin^2\theta = 1 - \cos^2\theta$, we get

$$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$$

The second part follows from a couple of simple observations. Namely, we may replace 3 by n in the equation above, and this will mean that we can write $\cos n\theta$ as a polynomial in $\cos\theta$ and $\sin\theta$. However, the $\sin\theta$ terms only show up because $i^2 = -1$, and hence their powers are always *even*. Thus we may always use the identity $\sin^2\theta = 1 - \cos^2\theta$ to reduce this polynomial to one in $\cos\theta$ alone.

What this tells us is that if we have a circuit that can multiply and add, that we can produce any harmonic of a signal from the given one.

The Chebyshev polynomials are a special class of polynomials that show up in many different areas of mathematics such as approximation theory, special function theory etc., and made their first appearance in 1854.