Amat433 Fall 2003 Final Exam Solutions

1. $$\frac{e^{2iz}}{z^4-1} = \frac{e^{2iz}}{(z+1)(z-1)(z+i)(z-i)} \equiv F(z)$$

$$\operatorname*{Res}_{z=1} F = \frac{e^{2i}}{2\cdot 2} = \frac{1}{4}e^{2i}$$

$$\operatorname*{Res}_{z=-1} F = \frac{e^{-2i}}{(-2)\cdot 2} = -\frac{1}{4}e^{-2i}$$

$$\operatorname*{Res}_{z=i} F = \frac{e^{-2}}{(-2)(2i)} = \frac{1}{4}i\,e^{-2}$$

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^4-1}\,dx = 2\pi i\left[\frac{1}{4}ie^{-2}\right] + \pi i\left[\frac{e^{2i}}{4} - \frac{e^{-2i}}{4}\right]$$

$$= \frac{-\pi}{2}e^{-2} - \frac{\pi}{2}\sin 2$$

$$\int_{-\infty}^{\infty} \frac{\cos 2x\,dx}{x^4-1} = -\frac{\pi}{2}\left[e^{-2}+\sin 2\right]. \leftarrow$$

2. $$I = \int_C \frac{\sin 2z}{(z+i)^3} = 2\pi i \left.\frac{f''(z)}{2!}\right|_{z=-i} \quad f(z) = \sin 2z$$

$$f''(z) = -4\sin 2z$$

$$I = \pi i(-4)\sin(-2i) = 4\pi i \sin 2i$$

$$I = 4\pi i\,\frac{1}{2i}\left(e^{-2}-e^{2}\right) = -2\pi\left(e^{2}-e^{-2}\right). \leftarrow$$

$$= -4\pi \sinh 2 \quad \leftarrow$$

3. $f(z) = z+5+\dfrac{z+e^{z}}{z-1}$

$$= (z-1)+6+\frac{z-1+1+e^{z-1+1}}{z-1}$$

$$= z-1+6+1+\frac{1}{z-1}+e\,\frac{e^{z-1}}{z-1}$$

$$= z-1+7+\frac{1}{z-1}+\frac{e}{z-1}\left[1+z-1+\frac{(z-1)^2}{2!}+\cdots\right]$$

$$= \frac{e+1}{z-1}+7+e+\left(1+\frac{e}{2}\right)(z-1)+e\sum_{n=2}^{\infty}\frac{(z-1)^{n-1}}{n!}$$

$b_1 = e+1 \qquad b_n = 0 \quad n \geq 2$

$a_0 = 7+e \qquad a_1 = 1+\frac{e}{2} \qquad a_n = \frac{e}{(n+1)!},\ n \geq 2.$

4. (a) $|A-\lambda I| = (1-\lambda)\begin{vmatrix} 1-\lambda & 1+i \\ 1-i & -\lambda \end{vmatrix} = (1-\lambda)[\lambda(\lambda-1)-2]$

$$= (1-\lambda)(\lambda^2-\lambda-2) = (1-\lambda)(\lambda-2)(\lambda+1)$$

$\lambda = -1, 1, 2 \leftarrow$

$$(A-\lambda I)v = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1+i \\ 0 & 1-i & -\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

$\lambda = -1 \quad V_{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1+i \end{pmatrix} \qquad \lambda = 1 \quad V_{(2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda = 2 \quad V_{(3)} = \begin{pmatrix} 0 \\ -2 \\ 1-i \end{pmatrix}$

(b) $-Ch(\lambda) = (\lambda-2)(\lambda^2-1) = \lambda^3-2\lambda^2-\lambda+2$

$$-Ch(A) = 0 = A^3-2A^2-A+2I$$

$$A^{-1} = \frac{1}{2}\left(-A^2+2A+I\right) = -\frac{1}{2}A^2+A+\frac{1}{2}I.$$

5. $\frac{X''}{X} = -\frac{Y''}{Y} = -\alpha^2$

$X = e^{i\alpha x}$ $\quad Y = \cosh \alpha y$

$U = \int_{-\infty}^{\infty} A(\alpha) e^{i\alpha x} \cosh \alpha y \, d\alpha$

$f(x) = \int_{-\infty}^{\infty} A(\alpha) e^{i\alpha x} \cosh \alpha b \, d\alpha$

$A \cosh \alpha b = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{2\pi}} e^{-i\alpha x} dx$

$= \frac{1}{2\pi} \int_{-a}^{a} e^{i\alpha x} dx = \frac{1}{2\pi} \left(e^{i\alpha a} - e^{-i\alpha a} \right) \frac{1}{i\alpha}$

$A \cosh \alpha b = \frac{1}{\pi \alpha} \sin \alpha a$

$A = \frac{1}{\pi \alpha} \frac{\sin \alpha a}{\cosh \alpha b}$

$U = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha \cosh \alpha b} e^{i\alpha x} \cosh \alpha y \, d\alpha$

$U = + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} \cos \alpha x \frac{\cosh \alpha y}{\cosh \alpha b} d\alpha \leftarrow$

(a) $L(H(t-a)) = \int_a^\infty e^{-pt}\,dt = \dfrac{e^{-pa}}{p} \qquad p>0$

$L(e^{\beta t}H(t)) = \int_0^\infty e^{(\beta-p)t}\,dt = \dfrac{1}{p-\beta} \qquad p>\beta$

(b) $q' + 2q = V_0 H(t-a) \qquad q(0)=0$

$pF + 2F = V_0 \dfrac{e^{-pa}}{p}$

$F = V_0 \dfrac{e^{-pa}}{(p+2)p} = \dfrac{V_0 e^{-pa}}{2}\left[\dfrac{1}{p} - \dfrac{1}{p+2}\right]$

$q = \dfrac{V_0}{2}\left[L^{-1}\left(\dfrac{e^{-pa}}{p}\right) - L^{-1}\dfrac{e^{-pa}}{p+2}\right]$

$q = \dfrac{V_0}{2}\left[H(t-a) - e^{-2(t-a)}H(t-a)\right].$